

On the Construction of Correlation Functions for the Integrable Supersymmetric Fermion Models

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We review the recent progress on the construction of the determinant representations of the correlation functions for the integrable supersymmetric fermion models. The factorizing F -matrices (or the so-called F -basis) play an important role in the construction. In the F -basis, the creation (and the annihilation) operators and the Bethe states of the integrable models are given in completely symmetric forms. This leads to the determinant representations of the scalar products of the Bethe states for the models. Based on the scalar products, the determinant representations of the correlation functions may be obtained. As an example, in this review, we give the determinant representations of the two-point correlation function for the $U_q(gl(2|1))$ (i.e. q -deformed) supersymmetric t - J model. The determinant representations are useful for analysing physical properties of the integrable models in the thermodynamical limit.

Keywords: Correlation functions; Drinfeld twists; Integrable supersymmetric fermion models

1. Introduction

It is well known that quantum integrable and exactly solvable systems based on the Yang-Baxter equation (YBE) play an important role in modern mathematics and physics. They have important applications in a startling variety of physical theories, such as the theory of the ultrasmall metallic grains (see e.g. refs.^{1,2} and references therein), the ($\mathcal{N} = 4$) four dimensional super-symmetric Yang-Mills gauge theories (see e.g. refs.^{3,4,5} and references therein), and string theories (see e.g. refs.^{6,7} and references therein).

In dealing with the integrable systems, the algebraic Bethe ansatz or the Quantum Inverse Scattering Method (QISM) provides a powerful tool to the diagonalization of their Hamiltonians. In this approach, Bethe states are constructed by the pseudo-particle creation operators which are from the off-diagonal entries of the monodromy matrix. After obtaining the eigenvalues of a system, one of the most interesting and challenging problems is to construct scalar products (including the norms) of the eigenstates and correlation functions^{8,9}. In 1981, Gaudin et al proposed a hypothesis that the norm of the coordinate eigenstates for the Heisenberg XXZ spin chain model is given by some Jacobians¹⁰. This hypothesis was proved completely by Korepin in¹¹. Moreover based on the results of this work, the authors in^{12,13,14} computed the correlation functions for the integrable XXX and XXZ models as well as the one-dimensional Bose Gas system. With the help of the auxiliary dual quantum fields¹⁵ and the determinant representation for the partition function of the six-vertex model with domain wall boundary conditions^{16,17}, the determinant representations of the correlation functions of the XXX and XXZ models were obtained^{8,18}. Let us remark that under the hypothesis concerning the space of physical states¹⁹, integral representations of correlation functions for integrable models on an infinite 1-d lattice can be obtained by using the technique of the q -deformed vertex operator^{20–26}.

In 1996, Maillet et al²⁷ proved that for the R -matrices of the Heisenberg XXX and XXZ spin chain systems, there exist non-degenerate lower-triangular F -matrices (i.e. the Drinfeld twists)²⁸ with which the R -matrices are factorized

$$R_{12}(\lambda_1, \lambda_2) = F_{21}^{-1}(\lambda_2, \lambda_1)F_{12}(\lambda_1, \lambda_2), \quad (1.1)$$

where $R \in \text{End}(V \otimes V)$ with V being the 2-dimensional $gl(2)$ or $U_q(gl(2))$ module. Working in the basis provided by the N -site F -matrix, i.e. the so-called F -basis, they proved that the entries of the monodromy matrix and therefore the Bethe states of the systems are simplified to take completely symmetric forms. This observation implies that the exact evaluation of scalar products and correlation functions of an integrable system is feasible within the framework of the algebraic Bethe ansatz. In²⁹, Kitanine et al obtained the determinant representation of the correlation functions for the XXX and XXZ models, and showed the scalar products and norms of the eigenstates of the systems obtained using the Drinfeld twist approach coincide with those obtained in^{10,11,16}.

The Drinfeld twist approach in ^{27,29} was generalized to other cases. In ³⁰, the Drinfeld twists associated with any finite-dimensional irreducible representations of the Yangian $Y[gl(2)]$ were investigated, and in ³¹ the correlation functions for the higher spin XXX chains were computed. In ³², the spontaneous magnetization of the XXZ chain on the finite lattice was represented. In ³³, Albert et al constructed the F -matrix of the $gl(m)$ rational Heisenberg model and obtained a polarization free representation of the creation operators. Using these results, they resolved the hierarchy of the nested Bethe ansatz for the $gl(m)$ model. In ^{34,35}, the Drinfeld twists of the elliptic XYZ model and Belavin model were constructed. In ³⁶, we obtained the determinant representations for the $U_q(gl(1|1))$ free fermion model.

As far as we know, the determinant representations of correlation functions were known only for integrable models related to $gl(2)$ algebra, and it had been a major longstanding problem to find the determinant representations of correlation functions for integrable models related to $gl(3)$ or other higher rank algebras. Very recently in ³⁷, based on our results on the Drinfeld twists and symmetric Bethe states ^{38,39,40}, we have presented a solution to this problem for $gl(2|1)$ algebra which is a graded version of $gl(3)$, and found the determinant representations of the correlation functions for the supersymmetric t - J model.

In this article, we review the recent progress that we made on the Drinfeld twists and on the determinant representations of the correlation functions for supersymmetric integrable models such as the supersymmetric t - J model. Supersymmetric integrable models form an important class of exactly soluble models ^{41–43} as they provide strongly correlated fermion systems of superconductivity ^{44–46} and have important applications in the AdS/CFT correspondence. In section 2, we briefly review the background of the integrable models and introduce the N -site R -matrix. In section 3, we discuss the properties of the monodromy matrices, and give the recursive relations as well as the representations of the local generators in terms of monodromy matrix elements. In section 4, we derive the factorizing F -matrix and its inverse. Then in section 5, we discuss the monodromy matrices in the F -basis. In section 6, we describe the construction of the determinant representations of scalar products and correlation functions, using the q -deformed supersymmetric t - J model as an example. In section 7, we conclude the review by offering some discussions and outlooks.

2. Integrable $U_q(gl(m|n))$ supersymmetric model

We first introduce some useful properties of the quantum superalgebra $U_q(gl(m|n))$. For more details, see refs. ^{40,47}. Let us fix two non-negative integers n, m such that $m+n \geq 2$ and a positive integer $N (\geq 2)$, and a generic complex number η such that the q -deformation parameter, related to η through $q = e^\eta$, is not a root of unity. Let V be a \mathbb{Z}_2 -graded $(m+n)$ -dimensional vector space with the orthonormal basis $\{|i\rangle, i = 1, \dots, m+n\}$. The \mathbb{Z}_2 -grading is chosen as: $[1] = \dots = [m] = 1, [m+1] = \dots = [m+n] = 0$.

The quantum superalgebra $U_q(gl(m|n))$ is a \mathbb{Z}_2 -graded unital associative superalgebra generated by the Cartan generators $E^{i,i}$, ($i = 1, \dots, m+n$) and the simple root generators $E^{j,j+1}$, $E^{j+1,j}$ ($j = 1, \dots, m+n-1$) with the \mathbb{Z}_2 -grading $[E^{i,i}] = 0$, $[E^{j+1,j}] = [E^{j,j+1}] = [j] + [j+1]$. The \mathbb{Z}_2 -graded vector space V supplies the fundamental $U_q(gl(m|n))$ -module and the generators of $U_q(gl(m|n))$ are represented in this space by $\pi(E^{i,i}) = e_{i,i}$, $\pi(E^{j,j+1}) = e_{j,j+1}$, $\pi(E^{j+1,j}) = e_{j+1,j}$, where $e_{i,j} \in \text{End}(V)$ is the elementary matrix with elements $(e_{i,j})_k^l = \delta_{jk} \delta_{il}$.

With the help the simple root generators, we can construct the non-simple root generators as follows

$$E^{\alpha,\gamma} = E^{\alpha,\beta} E^{\beta,\gamma} - q^{-(-1)^{[\beta]}} E^{\beta,\gamma} E^{\alpha,\beta}, \quad 1 \leq \alpha < \beta < \gamma \leq m+n, \quad (2.1)$$

$$E^{\gamma,\alpha} = E^{\gamma,\beta} E^{\beta,\alpha} - q^{(-1)^{[\beta]}} E^{\beta,\alpha} E^{\gamma,\beta}, \quad 1 \leq \alpha < \beta < \gamma \leq m+n. \quad (2.2)$$

$U_q(gl(m|n))$ is a \mathbb{Z}_2 -graded triangular Hopf superalgebra. For the Cartan and simple generators, the coproduct $\Delta : U_q(gl(m|n)) \rightarrow U_q(gl(m|n)) \otimes U_q(gl(m|n))$ is defined by

$$\Delta(E^{i,i}) = 1 \otimes E^{i,i} + E^{i,i} \otimes 1, \quad i = 1, \dots, m+n, \quad (2.3)$$

$$\Delta(E^{j,j+1}) = 1 \otimes E^{j,j+1} + E^{j,j+1} \otimes q^{h^j}, \quad (2.4)$$

$$\Delta(E^{j+1,j}) = q^{-h^j} \otimes E^{j+1,j} + E^{j+1,j} \otimes 1, \quad (2.5)$$

where $h^j = (-1)^{[j]} E^{j,j} - (-1)^{[j+1]} E^{j+1,j+1}$. Throughout, we will use the following notation:

$$E_{\alpha,\beta} \equiv \Delta^{(N-1)}(E^{\alpha,\beta}) = (\text{id} \otimes \Delta^{(N-2)})\Delta(E^{\alpha,\beta}), \quad (2.6)$$

for any generator $E^{\alpha,\beta}$ ($\alpha, \beta = 1, \dots, m+n$) of $U_q(gl(m|n))$.

2.1. Integrability of the model

The R -matrix, $R \in \text{End}(V \otimes V)$, depends on the difference of two spectral parameters λ_1 and λ_2 associated with the two copies of V , and is, in this grading, given by ^{41,42,43}

$$\begin{aligned} R_{12}(\lambda_1, \lambda_2) &= R_{12}(\lambda_1 - \lambda_2) \\ &= c_{12} \sum_{i=1}^m e_{i,i} \otimes e_{i,i} + \sum_{i=m+1}^{m+n} e_{i,i} \otimes e_{i,i} + a_{12} \sum_{i \neq j=1}^{m+n} e_{i,i} \otimes e_{j,j} \\ &\quad + b_{12}^- \sum_{i>j=1}^{m+n} (-1)^{[j]} e_{i,j} \otimes e_{j,i} + b_{12}^+ \sum_{j>i=1}^{m+n} (-1)^{[j]} e_{i,j} \otimes e_{j,i}, \end{aligned} \quad (2.7)$$

where

$$\begin{aligned} a_{12} = a(\lambda_1, \lambda_2) &\equiv \frac{\sinh(\lambda_1 - \lambda_2)}{\sinh(\lambda_1 - \lambda_2 + \eta)}, \quad b_{12}^\pm = b^\pm(\lambda_1, \lambda_2) \equiv \frac{e^{\pm(\lambda_1 - \lambda_2)} \sinh \eta}{\sinh(\lambda_1 - \lambda_2 + \eta)}, \\ c_{12} = c(\lambda_1, \lambda_2) &\equiv \frac{\sinh(\lambda_1 - \lambda_2 - \eta)}{\sinh(\lambda_1 - \lambda_2 + \eta)}, \end{aligned} \quad (2.8)$$

and η is the so-called crossing parameter. One can easily check that the R -matrix satisfies the unitary relation $R_{21}R_{12} = 1$.

Let us introduce the $(N + 1)$ -fold tensor product space $V^{\otimes N+1}$, whose components are labelled by $0, 1, \dots, N$ from the left to the right. As usual, the 0-th space, denoted by V_0 (V_i for the i -th space), corresponds to the auxiliary space and the other N spaces constitute the quantum space $V^{\otimes N}$. Moreover, for each factor space V_i , $i = 0, \dots, N$, we associate a complex parameter ξ_i . The parameter associated with the 0-th space is usually called the *spectral* parameter which is set to $\xi_0 = \lambda$ in this paper, and the other parameters are called the *inhomogeneous* parameters. In this paper we always assume that all the complex parameters u and $\{\xi_i | i = 1, \dots, N\}$ be *generic* ones. Hereafter we adopt the standard notation: for any matrix $A \in \text{End}(V)$, A_j (or $A_{(j)}$) is an embedding operator in the tensor product space, which acts as A on the j -th space and as an identity on the other factor spaces; $R_{ij} = R_{ij}(\xi_i, \xi_j)$ is an embedding operator of R -matrix in the tensor product space, which acts as an identity on the factor spaces except for the i -th and j -th ones.

The R -matrix satisfies the graded Yang-Baxter equation (GYBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \quad (2.9)$$

In terms of the matrix elements defined by

$$R(\lambda)(v^{i'} \otimes v^{j'}) = \sum_{i,j} R(\lambda)_{ij}^{i'j'} (v^i \otimes v^j), \quad (2.10)$$

the GYBE reads

$$\begin{aligned} & \sum_{i',j',k'} R(\lambda_1 - \lambda_2)_{ij}^{i'j'} R(\lambda_1 - \lambda_3)_{i'k}^{i''k'} R(\lambda_2 - \lambda_3)_{j'k'}^{j''k''} (-1)^{[j']([i']+[i''])} \\ &= \sum_{i',j',k'} R(\lambda_2 - \lambda_3)_{jk}^{j'k'} R(\lambda_1 - \lambda_3)_{ik}^{i'k''} R(\lambda_1 - \lambda_2)_{i'j'}^{i''j''} (-1)^{[j']([i]+[i'])}. \end{aligned} \quad (2.11)$$

The quantum monodromy matrix $T(\lambda)$ of the supersymmetric chain of length N is defined as

$$T(\lambda) = R_{0N}(\lambda, \xi_N) R_{0N-1}(\lambda, \xi_{N-1}) \dots R_{01}(\lambda, \xi_1), \quad (2.12)$$

where the index 0 refers to the auxiliary space and $\{\xi_i\}$ are arbitrary inhomogeneous parameters depending on site i .

By the GYBE, one may prove that the monodromy matrix satisfies the GYBE

$$R_{00'}(\lambda - \mu) T_0(\lambda) T_{0'}(\mu) = T_{0'}(\mu) T_0(\lambda) R_{00'}(\lambda - \mu). \quad (2.13)$$

Define the transfer matrix $t(\lambda) = \text{str}_0 T(\lambda)$, where str_0 denotes the supertrace over the auxiliary space. Then the Hamiltonian of our model is given by $H = d \ln t(\lambda) / d\lambda|_{\lambda=0}$. This model is integrable thanks to the commutativity of the transfer matrix for different parameters, $[t(\lambda), t(\mu)] = 0$, which can be verified by using the GYBE.

2.2. N -site R -matrices

Let σ be an element of the permutation group \mathcal{S}_{N+1} . We generalize the R -matrix (2.7) to the N -site R -matrix with the help of σ as follows. The concept of N -site R -matrices was first introduced in ^{27,33}.

Definition 1. One can define a mapping from \mathcal{S}_{N+1} to $\text{End}(V_0 \otimes \mathcal{H})$ which associate in a unique way an element $R_{0\dots N}^\sigma \in \text{End}(V_0 \otimes \mathcal{H})$ to any element σ of the permutation group \mathcal{S}_{N+1} . The mapping has the following composition law

$$R_{0\dots N}^{\sigma\sigma'} = \mathcal{P}^\sigma R_{0\dots N}^{\sigma'} (\mathcal{P}^\sigma)^{-1} R_{0\dots N}^\sigma = R_{\sigma(0)\dots\sigma(N)}^{\sigma'} R_{0\dots N}^\sigma, \quad \forall \sigma, \sigma' \in \mathcal{S}_{N+1}, \quad (2.14)$$

where \mathcal{P}^σ is the \mathbb{Z}_2 -graded permutation operator

$$\mathcal{P}^\sigma |i_0\rangle_{(0)} \dots |i_N\rangle_{(N)} = |i_0\rangle_{(\sigma(0))} \dots |i_N\rangle_{(\sigma(N))}.$$

For any elementary permutation σ_j with $\sigma_j(j, j+1) = (j+1, j)$, $j = 0, \dots, N$, $R_{0\dots N}^{\sigma_j} = R_{j\ j+1}$.

From the definition, one may prove the following properties of the map $R_{0\dots N}^\sigma$:

- Uniqueness. For any element $\sigma \in \mathcal{S}_{N+1}$, the corresponding $R_{0\dots N}^\sigma$ can be constructed through (2.14) as follows. Let σ be decomposed in a minimal way in terms of elementary permutation as $\sigma = \sigma_{\beta_1} \dots \sigma_{\beta_p}$ where the positive integer p is the length of σ . The composition law enables one to obtain the expression of the associated $R_{0\dots N}^\sigma$. The GYBE and the unitary relation guarantee the uniqueness of $R_{0\dots N}^\sigma$.
- $R_{0\dots N}^{id} = id$. By using the unitary relation $R_{ij}R_{ji} = 1$, this property can be easily proved.
- For the cyclic permutation $\sigma'_c = \sigma_0\sigma_1 \dots \sigma_N$ of the group \mathcal{S}_{N+1} , the N -site R -matrix $R_{0\dots N}^{\sigma'_c}$ gives the monodromy matrix $T(\lambda)$ of the $U_q(gl(m|n))$ spin chain on an N -site lattice:

$$R_{0\dots N}^{\sigma'_c} = R_{0\ N} R_{01\ N-1} \dots R_{01} = T(\lambda) \equiv T_0(\lambda) = T_{0,1\dots N}(\lambda). \quad (2.15)$$

- For any $\sigma \in \mathcal{S}_N$ which acts on the quantum space \mathcal{H} , by using the GYBE, one may prove

$$R_{1\dots N}^\sigma T_{0,1\dots N} = T_{0,\sigma'(1\dots N)} R_{1\dots N}^\sigma. \quad (2.16)$$

Moreover, let $\sigma_c = \sigma_1 \dots \sigma_{N-1}$ be the cyclic permutation. We have

$$R_{1\dots N}^{\sigma_c^k} T_{0,1\dots N}(\lambda) = T_{0,k+1\dots N1\dots k}(\lambda) R_{1\dots N}^{\sigma_c^k}. \quad (2.17)$$

3. Some properties of the monodromy matrix elements

3.1. Recursive relations

The monodromy matrix $T(\lambda)$ (2.12) may be decomposed in terms of the basis of $\text{End}(V_0)$ as

$$T(\lambda) = \sum_{i,j=1}^{m+n} T_{i,j}(\lambda) E_{(0)}^{i,j} \equiv \sum_{i,j=1}^{m+n} T_{i,j}(\lambda) e_{i,j}, \quad (3.1)$$

where the matrix elements $T_{i,j}(\lambda)$ are operators acting on the quantum space \mathcal{H} and have the \mathbb{Z}_2 -grading: $[T_{i,j}(\lambda)] = [e_{i,j}] = [i] + [j]$.

For the entries $T_{m+n,m+n-l}(\lambda)$ of the monodromy matrix, we have the following theorem ⁴⁰:

Theorem 1. *The matrix elements $T_{m+n,m+n-l}(\lambda)$ and $T_{m+n-l,m+n}(\lambda)$ ($l = 1, \dots, m+n-1$) of the monodromy matrix can be expressed in terms of $T_{m+n,m+n}(\lambda)$ and the generators of $U_q(\mathfrak{gl}(m|n))$ by the following recursive relations:*

$$\begin{aligned} T_{m+n,m+n-l}(\lambda) &= \left(q^{-(-1)^{[m+n]}} E_{m+n-l,m+n} T_{m+n,m+n}(\lambda) \right. \\ &\quad \left. - T_{m+n,m+n}(\lambda) E_{m+n-l,m+n} \right) q^{-\sum_{k=1}^l H_{m+n-k}} \\ &\quad - \sum_{\alpha=1}^{l-1} (1 - q^{-2(-1)^{[m+n-\alpha]}}) T_{m+n,m+n-\alpha}(\lambda) E_{m+n-l,m+n-\alpha} q^{-\sum_{k=\alpha+1}^l H_{m+n-k}}. \end{aligned} \quad (3.2)$$

$$\begin{aligned} T_{m+n-l,m+n}(\lambda) &= (-1)^{[m+n]+[m+n-l]} q^{\sum_{k=1}^l H_{m+n-k}} \\ &\quad \times \left(q^{(-1)^{[m+n]}} T_{m+n,m+n}(\lambda) E_{m+n,m+n-l} - E_{m+n,m+n-l} T_{m+n,m+n}(\lambda) \right) \\ &\quad - \sum_{\alpha=1}^{l-1} (-1)^{([m+n]+[m+n-l])([m+n-l]+[m+n-l+\alpha])} (1 - q^{2(-1)^{[m+n-l+\alpha]}}) \\ &\quad \times q^{\sum_{k=l-\alpha+1}^l H_{m+n-k}} E_{m+n-l+\alpha,m+n-l} T_{m+n-l+\alpha,m+n}(\lambda), \end{aligned} \quad (3.3)$$

where $E_{i,j}$ is the N -site $U_q(\mathfrak{gl}(m|n))$ generator which is given by local generator $E_{(k)}^{i,j}$ with the help of (2.6), and $H_j = (-1)^{[j]} E_{j,j} - (-1)^{[j+1]} E_{j+1,j+1}$ ($j = 1, \dots, m+n-1$).

We call the second terms in the R.H.S. of (3.2) and (3.3) *quantum correction term*, which vanishes in the rational limit ($q \rightarrow 1$). Moreover, such a nontrivial correction term only occurs in the higher rank models (i.e., when $m+n \geq 3$). In the rational limit: $q \rightarrow 1$, (3.2) and (3.3) reduces to the (anti)commutation relations used in ^{33,39}. The detailed proof for this theorem may be found in ⁴⁰. Here we give two examples to illustrate the theorem.

8 *Shao-You Zhao et al*

- For $m = 2, n = 0$, i.e. the $U_q(gl(2|0))$ model:

$$T_{2,1} = [qE_{1,2}T_{2,2} - T_{2,2}E_{1,2}]q^{H_1}, \quad (3.4)$$

$$T_{1,2} = q^{H_1}[q^{-1}T_{2,2}E_{2,1} - E_{2,1}T_{2,2}]. \quad (3.5)$$

- For $m = 2, n = 1$, i.e. the q -deformed supersymmetric $t - J$ model:

$$T_{3,2}(\lambda) = [q^{-1}E_{2,3}T_{3,3}(\lambda) - T_{3,3}(\lambda)E_{2,3}]q^{-H_2}, \quad (3.6)$$

$$T_{2,3}(\lambda) = -q^{H_2}[qT_{3,3}(\lambda)E_{3,2} - E_{3,2}T_{3,3}(\lambda)], \quad (3.7)$$

$$T_{3,1}(\lambda) = [q^{-1}E_{1,3}T_{3,3}(\lambda) - T_{3,3}(\lambda)E_{1,3}]q^{-H_2-H_1} \\ - (1 - q^2)T_{3,2}(\lambda)E_{1,2}q^{-H_1}, \quad (3.8)$$

$$T_{1,3}(\lambda) = -q^{H_1+H_2}[qT_{3,3}(\lambda)E_{3,1} - E_{3,1}T_{3,3}(\lambda)] \\ - (1 - q^{-2})q^{H_1}E_{2,1}T_{2,3}(\lambda). \quad (3.9)$$

3.2. Representation of the local operators

In ²⁹, Kitanine et al constructed the local spin operators of the inhomogeneous spin-1/2 XXX and XXZ Heisenberg chains in terms of the corresponding monodromy matrix elements. The results were generalized to more general cases in ^{48,49}.

Theorem 2. *For the monodromy matrix constructed by the $U_q(gl(m|n))$ R -matrix which is a solution of the GYBE and satisfies the properties: i.) Regularity. $R_{ij}(\lambda_i, \lambda_j = \lambda_i) = \mathcal{P}_{ij}$ and ii.) $R(\lambda, \mu)_{ij}^{i'j'} = (-1)^{[i]+[j]+[i']+[j']}R(\lambda, \mu)_{i'j'}^{ij}$, the local generator $E_{(\kappa)}^{\alpha, \beta}$, which acts on the given site κ , can be represented in terms of the entries of the monodromy matrix as the following formula,*

$$E_{(\kappa)}^{\alpha, \beta} = (-1)^{[\alpha]+[\beta]} \prod_{j=1}^{\kappa-1} str_0(T_{0,1\dots N}(\xi_j)) T_{\alpha, \beta}(\xi_\kappa) \prod_{j=\kappa+1}^N str_0(T_{0,1\dots N}(\xi_j)). \quad (3.10)$$

Here \mathcal{P} is the superpermutation operator, i.e. $\mathcal{P}(|x\rangle \otimes |y\rangle) = (-1)^{[x][y]}(|y\rangle \otimes |x\rangle)$. In terms of the generators $E_{(\kappa)}^{\alpha, \beta}$, \mathcal{P}_{ij} can be written by $\mathcal{P}_{ij} = \sum_{\alpha, \beta} (-1)^{[\alpha]} E_{(i)}^{\alpha, \beta} E_{(j)}^{\beta, \alpha}$.

To prove this theorem, one considers the supertrace $str_0(X_0 T_{0, \kappa \dots N 1 \dots \kappa-1}(\xi_\kappa))$, where $X_0 \in U_q(gl(m|n))$ and $X_0 = \sum_{\alpha, \beta=1}^{m+n} X_{\alpha, \beta} E_{(0)}^{\beta, \alpha}$,

$$\begin{aligned} & str_0(X_0 T_{0, \kappa \dots N 1 \dots \kappa-1}(\xi_\kappa)) \\ &= str_0(X_0 R_{0\kappa-1}(\xi_n, \xi_{\kappa-1}) \dots R_{01}(\xi_n, \xi_1)) R_{0N}(\xi_n, \xi_N) \dots R_{0\kappa+1}(\xi_n, \xi_{\kappa+1}) \mathcal{P}_{0\kappa}) \\ &= str_0(X_0 \mathcal{P}_{0\kappa}) R_{\kappa\kappa-1} \dots R_{\kappa 1} R_{\kappa N} \dots R_{\kappa\kappa+1} \\ &= \sum_{\alpha, \beta=1}^{m+n} (-1)^{[\alpha]+[\beta]} X_{\alpha, \beta} E_{(\kappa)}^{\beta, \alpha} R_{\sigma_c^{\kappa-1}(1\dots N)}^{\sigma_c}. \end{aligned} \quad (3.11)$$

where $\sigma_c = \sigma_1 \dots \sigma_{N-1}$ ($\sigma \in \mathcal{S}_N$) and in the derivation of (3.11), we have used the decomposition law (2.14).

Similarly, one may prove the following useful relations:

$$\begin{aligned} \text{str}_0(T_{0,1\dots N}(\xi_\kappa)) &= R_{\sigma_c^{\kappa-1}(1\dots N)}^{\sigma_c}, \quad \prod_{j=1}^{\kappa} \text{str}_0(T_{0,1\dots N}(\xi_j)) = R_{1\dots N}^{\sigma_c^\kappa}, \\ \prod_{j=\kappa+1}^N \text{str}_0(T_{0,1\dots N}(\xi_j)) &= \left(R_{1\dots N}^{\sigma_c^\kappa}\right)^{-1}. \end{aligned} \quad (3.12)$$

On the other hand, with the help of the decomposition laws (2.14), (2.17) and (3.12), we have

$$\begin{aligned} &\text{str}_0(X_0 T_{0,\kappa\dots N 1\dots \kappa-1}(\xi_\kappa)) \\ &= \text{str}_0\left(X_0 T_{0,\kappa\dots N 1\dots \kappa-1}(\xi_\kappa) R_{1\dots N}^{\sigma_c^{\kappa-1}} \left(R_{1\dots N}^{\sigma_c^{\kappa-1}}\right)^{-1}\right) \\ &= R_{1\dots N}^{\sigma_c^{\kappa-1}} \cdot \text{str}_0(X_0 T_{0,1\dots N}(\xi_\kappa)) \cdot \left(R_{1\dots N}^{\sigma_c^{\kappa-1}}\right)^{-1} \\ &= (-1)^{[\alpha]} \prod_{j=1}^{\kappa} \text{str}_0(T_{0,1\dots N}(\xi_j)) \cdot T_{\alpha,\beta} \cdot \prod_{j=\kappa}^N \text{str}_0(T_{0,1\dots N}(\xi_j)). \end{aligned} \quad (3.13)$$

Then comparing (3.11) with (3.13), and considering (3.12) and the supertranspose property $A_{ij} = (-1)^{[j]([i]+[j])} A_{ji}$, one arrives at (3.10),

4. Factorizing F-matrices and their inverses

In ²⁷, Maillet *et al* found that the R -matrices for the XXX and XXZ Heisenberg spin chain models are factorized in terms of the F -matrices. The results were generalized to $gl(n)$ spin chain system by Albert et al ³³, where the authors constructed the factorizing F -matrices (Drinfeld twists) explicitly on the N -fold tensor product space.

Let \mathcal{S}_N be the permutation group associated with the indices $(1, \dots, N)$ and $R_{1\dots N}^\sigma$ the N -site R -matrix associated with $\sigma \in \mathcal{S}_N$. $R_{1\dots N}^\sigma$ acts non-trivially on the quantum space \mathcal{H} and trivially (i.e as an identity) on the auxiliary space.

Definition 2. The N -site F -matrix $F_{1\dots N}(\xi_1, \dots, \xi_N)$ is an operator in $\text{End}(\mathcal{H})$ and satisfies the following three properties: I.) lower-triangularity; II.) non-degeneracy; III.) factorization, namely,

$$F_{\sigma(1)\dots\sigma(N)}(\xi_{\sigma(1)}, \dots, \xi_{\sigma(N)}) R_{1\dots N}^\sigma = F_{1\dots N}(\xi_1, \dots, \xi_N), \quad \forall \sigma \in \mathcal{S}_N. \quad (4.1)$$

Proposition 1. The N -site factorizing F -matrix for the $U_q(gl(m|n))$ supersymmetric model, given by

$$F_{1\dots N} \equiv F_{1\dots N}(\xi_1, \dots, \xi_N) = \sum_{\sigma \in \mathcal{S}_N} \sum_{\alpha_{\sigma(1)} \dots \alpha_{\sigma(N)}=1}^{m+n} \prod_{j=1}^N P_{\sigma(j)}^{\alpha_{\sigma(j)}} S(\sigma, \alpha_\sigma) R_{1\dots N}^\sigma, \quad (4.2)$$

satisfies the properties I, II and III in the definition 2.

Here, P_i^α is the embedding of the project operator P^α in the i -th space with $(P^\alpha)_{kl} = \delta_{kl}\delta_{k\alpha}$, $S(\sigma, \alpha_\sigma)$ is a c-number function of σ, α_σ and the element c_{ij} of the R-matrix

$$S(\sigma, \alpha_\sigma) \equiv \exp \left\{ \frac{1}{2} \sum_{l>k=1}^N \left(1 - (-1)^{[\alpha_{\sigma(k)}]} \right) \delta_{\alpha_{\sigma(k)}, \alpha_{\sigma(l)}} \ln(1 + c_{\sigma(k)\sigma(l)}) \right\} \quad (4.3)$$

and the sum \sum^* is defined by

$$\alpha_{\sigma(i+1)} \geq \alpha_{\sigma(i)} (\sigma(i+1) > \sigma(i)); \quad \alpha_{\sigma(i+1)} > \alpha_{\sigma(i)} (\sigma(i+1) < \sigma(i)). \quad (4.4)$$

Here we outline the proof given in ^{38,40}. The definition of $F_{1\dots N}$ (4.2) and the summation condition (4.4) imply that $F_{1\dots N}$ is a lower-triangular matrix. Moreover, one can easily check that the F -matrix is non-degenerate because all diagonal elements are non-zero.

We now prove that the F -matrix (4.2) satisfies the property III. Any given permutation $\sigma \in \mathcal{S}_N$ can be decomposed into elementary ones of the group \mathcal{S}_N as $\sigma = \sigma_{i_1} \dots \sigma_{i_k}$. By (2.14), we have, if the property III holds for any elementary permutation σ_i ,

$$\begin{aligned} F_{\sigma(1\dots N)} R_{1\dots N}^\sigma &= F_{\sigma_{i_1} \dots \sigma_{i_k} (1\dots N)} R_{\sigma_{i_1} \dots \sigma_{i_{k-1}} (1\dots N)}^{\sigma_{i_k}} R_{\sigma_{i_1} \dots \sigma_{i_{k-2}} (1\dots N)}^{\sigma_{i_{k-1}}} \dots R_{1\dots N}^{\sigma_{i_1}} \\ &= F_{\sigma_{i_1} \dots \sigma_{i_{k-1}} (1\dots N)} R_{\sigma_{i_1} \dots \sigma_{i_{k-2}} (1\dots N)}^{\sigma_{i_{k-1}}} \dots R_{1\dots N}^{\sigma_{i_1}} = \dots = F_{\sigma_{i_1} (1\dots N)} R_{1\dots N}^{\sigma_{i_1}} = F_{1\dots N}. \end{aligned}$$

For the elementary permutation σ_i , we have

$$\begin{aligned} F_{\sigma_i(1\dots N)} R_{1\dots N}^{\sigma_i} &= \sum_{\sigma \in \mathcal{S}_N} \sum_{\alpha_{\sigma_i\sigma(1)} \dots \alpha_{\sigma_i\sigma(N)}}^* \prod_{j=1}^N P_{\sigma_i\sigma(j)}^{\alpha_{\sigma_i\sigma(j)}} S(\sigma_i\sigma, \alpha_{\sigma_i\sigma}) R_{\sigma_i(1\dots N)}^\sigma R_{1\dots N}^{\sigma_i} \\ &= \sum_{\sigma \in \mathcal{S}_N} \sum_{\alpha_{\sigma_i\sigma(1)} \dots \alpha_{\sigma_i\sigma(N)}}^* \prod_{j=1}^N P_{\sigma_i\sigma(j)}^{\alpha_{\sigma_i\sigma(j)}} S(\sigma_i\sigma, \alpha_{\sigma_i\sigma}) R_{1\dots N}^{\sigma_i\sigma} \\ &= \sum_{\tilde{\sigma} \in \mathcal{S}_N} \sum_{\alpha_{\tilde{\sigma}(1)} \dots \alpha_{\tilde{\sigma}(N)}}^{*(i)} \prod_{j=1}^N P_{\tilde{\sigma}(j)}^{\alpha_{\tilde{\sigma}(j)}} S(\tilde{\sigma}, \alpha_{\tilde{\sigma}}) R_{1\dots N}^{\tilde{\sigma}}, \end{aligned} \quad (4.5)$$

where $\tilde{\sigma} = \sigma_i\sigma$, and the summation sequences of $\alpha_{\tilde{\sigma}}$ in $\sum^{*(i)}$ now has the form

$$\alpha_{\tilde{\sigma}(j+1)} \geq \alpha_{\tilde{\sigma}(j)} (\sigma_i\tilde{\sigma}(j+1) > \sigma_i\tilde{\sigma}(j)); \quad \alpha_{\tilde{\sigma}(j+1)} > \alpha_{\tilde{\sigma}(j)} (\sigma_i\tilde{\sigma}(j+1) < \sigma_i\tilde{\sigma}(j)). \quad (4.6)$$

Comparing (4.6) with (4.4), we find that the only difference between them is the transposition σ_i factor in the “if” conditions. For a given $\tilde{\sigma} \in \mathcal{S}_N$ with $\tilde{\sigma}(j) = i$ and $\tilde{\sigma}(k) = i+1$, we now examine how the elementary transposition σ_i will affect the inequalities (4.6). If $|j-k| > 1$, then σ_i does not affect the sequence of $\alpha_{\tilde{\sigma}}$ at all, that is, the sign of inequality “ $>$ ” or “ \geq ” between two neighboring root indexes is unchanged with the action of σ_i . If $|j-k| = 1$, then in the summation sequences

of $\alpha_{\tilde{\sigma}}$, when $\tilde{\sigma}(j+1) = i+1$ and $\tilde{\sigma}(j) = i$, sign “ \geq ” changes to “ $>$ ”, while when $\tilde{\sigma}(j+1) = i$ and $\tilde{\sigma}(j) = i+1$, “ $>$ ” changes to “ \geq ”. Thus (4.4) and (4.5) differ only when equal labels $\alpha_{\tilde{\sigma}}$ appear. With the help of the relation $c_{21}c_{12} = 1$, we can prove that in this case the product $F_{\sigma_i(1\dots N)} R_{1\dots N}^{\sigma_i}$ still equals to $F_{1\dots N}$ (one sees in ³⁸ for detailed proof). Thus, we obtain

$$R_{1\dots N}^{\sigma}(\xi_1, \dots, \xi_N) = F_{\sigma(1\dots N)}^{-1}(\xi_{\sigma(1)}, \dots, \xi_{\sigma(N)}) F_{1\dots N}(\xi_1, \dots, \xi_N), \quad (4.7)$$

Therefore the factorizing F -matrix $F_{1\dots N}$ of $U_q(gl(m|n))$ is proved to satisfy all three properties.

From the expression of the F -matrix, one knows that it has an even grading, i.e., $[F_{1\dots N}] = 0$.

The non-degenerate property of the F -matrix implies that we can find the inverse matrix $F_{1\dots N}^{-1}$.

Proposition 2. *The inverse of the F -matrix is given by*

$$F_{1\dots N}^{-1} = F_{1\dots N}^* \prod_{i < j} \Delta_{ij}^{-1}, \quad (4.8)$$

where

$$F_{1\dots N}^* = \sum_{\sigma \in S_N} \sum_{\alpha_{\sigma(1)} \dots \alpha_{\sigma(N)} = 1}^{m+n} S(\sigma, \alpha_{\sigma}) R_{\sigma(1\dots N)}^{\sigma^{-1}} \prod_{j=1}^N P_{\sigma(j)}^{\alpha_{\sigma(j)}}, \quad (4.9)$$

and

$$[\Delta_{ij}]_{\alpha_i \alpha_j}^{\beta_i \beta_j} = \delta_{\alpha_i \beta_i} \delta_{\alpha_j \beta_j} \begin{cases} a_{ij} & \text{if } \alpha_i > \alpha_j \\ a_{ji} & \text{if } \alpha_i < \alpha_j, \\ 1 & \text{if } \alpha_i = \alpha_j = m+1, \dots, m+n, \\ (1+c_{ij})(1+c_{ji}) & \text{if } \alpha_i = \alpha_j = 1, \dots, m. \end{cases} \quad (4.10)$$

Here the sum \sum^{**} is taken over all possible α_i which satisfies the following non-increasing constraints:

$$\alpha_{\sigma(i+1)} \leq \alpha_{\sigma(i)} (\sigma(i+1) < \sigma(i)); \quad \alpha_{\sigma(i+1)} < \alpha_{\sigma(i)} (\sigma(i+1) > \sigma(i)). \quad (4.11)$$

We outline the proof in ^{38,40}. We compute the product of $F_{1\dots N}$ and $F_{1\dots N}^*$. Substituting (4.2) and (4.9) into the product, we have

$$\begin{aligned} F_{1\dots N} F_{1\dots N}^* &= \sum_{\sigma \in S_N} \sum_{\sigma' \in S_N} \sum_{\alpha_{\sigma_1} \dots \alpha_{\sigma_N}}^* \sum_{\beta_{\sigma'_1} \dots \beta_{\sigma'_N}}^{**} S(\sigma, \alpha_{\sigma}) S(\sigma', \beta_{\sigma'}) \\ &\times \prod_{i=1}^N P_{\sigma(i)}^{\alpha_{\sigma(i)}} R_{1\dots N}^{\sigma} R_{\sigma'(1\dots N)}^{\sigma'^{-1}} \prod_{i=1}^N P_{\sigma'(i)}^{\beta_{\sigma'(i)}} \\ &= \sum_{\sigma \in S_N} \sum_{\sigma' \in S_N} \sum_{\alpha_{\sigma_1} \dots \alpha_{\sigma_N}}^* \sum_{\beta_{\sigma'_1} \dots \beta_{\sigma'_N}}^{**} S(\sigma, \alpha_{\sigma}) S(\sigma', \beta_{\sigma'}) \prod_{i=1}^N P_{\sigma(i)}^{\alpha_{\sigma(i)}} R_{\sigma'(1\dots N)}^{\sigma'^{-1}\sigma} \prod_{i=1}^N P_{\sigma'(i)}^{\beta_{\sigma'(i)}}. \end{aligned}$$

To evaluate the R.H.S., we examine the matrix element of the R -matrix $\left(R_{\sigma'(1\dots N)}^{\sigma'^{-1}\sigma}\right)_{\beta_{\sigma'(N)}\dots\beta_{\sigma'(1)}}^{\alpha_{\sigma(N)}\dots\alpha_{\sigma(1)}}$. Note that the sequence $\{\alpha_\sigma\}$ is non-decreasing and $\{\beta_{\sigma'}\}$ is non-increasing. Thus the non-vanishing condition of this R -matrix element requires that α_σ and $\beta_{\sigma'}$ satisfy the relations $\beta_{\sigma'(N)} = \alpha_{\sigma(1)}, \dots, \beta_{\sigma'(1)} = \alpha_{\sigma(N)}$. Then by using the sum conditions (4.4), (4.11) and the existence condition of the elements of the elementary R -matrix $R(\lambda, \mu)_{ij}^{i'j'}$, i.e. $i+j = i'+j'$, one can verify³³ that the relations between the roots β and α are fulfilled only if $\sigma'(N) = \sigma(1), \dots, \sigma'(1) = \sigma(N)$. Let $\bar{\sigma}$ be the maximal element of the S_N which reverses the site labels $\bar{\sigma}(1, \dots, N) = (N, \dots, 1)$. Then we have $\sigma' = \sigma\bar{\sigma}$. Therefore we have

$$F_{1\dots N}F_{1\dots N}^* = \sum_{\sigma \in S_N} \sum_{\alpha_{\sigma(1)}\dots\alpha_{\sigma(N)}}^* S(\sigma, \alpha_\sigma) S(\sigma, \alpha_\sigma) \prod_{i=1}^N P_{\sigma(i)}^{\alpha_{\sigma(i)}} R_{\sigma(N\dots 1)}^{\bar{\sigma}} \prod_{i=1}^N P_{\sigma(i)}^{\alpha_{\sigma(i)}} \quad (4.12)$$

The decomposition of $R^{\bar{\sigma}}$ in terms of elementary R -matrices is unique module the GYBE. One reduces from (4.12) that FF^* is a diagonal matrix: $F_{1\dots N}F_{1\dots N}^* = \prod_{i < j} \Delta_{ij}$. Then (4.8) is a simple consequence of the above equation.

5. Monodromy matrix in the F -basis

In the previous section, we have given the F -matrix and its inverse which act on the quantum space \mathcal{H} . The non-degeneracy of the F -matrix means that its column vectors also form a complete basis of \mathcal{H} , which is called the F -basis. In this section, we study the generators of $U_q(gl(m|n))$ and the elements of the monodromy matrix in the F -basis.

Introduce the generators in the F -basis: $\tilde{E}_{i,j} = F_{1\dots N} E_{i,j} F_{1\dots N}^{-1}$, ($i, j = 1, \dots, m+n$). Then,

Theorem 3. *In the F -basis the Cartan and the simple generators of $U_q(gl(m|n))$ are given by*

$$\tilde{E}_{i,i} = E_{i,i} = \sum_{k=1}^N E_{(k)}^{i,i}, \quad i = 1, \dots, m+n, \quad (5.1)$$

$$\tilde{E}_{j,j+1} = \sum_{k=1}^N E_{(k)}^{j,j+1} \otimes_{\gamma \neq k} G_{(\gamma)}^{j,j+1}(k, \gamma), \quad j = 1, \dots, m+n-1, \quad (5.2)$$

$$\tilde{E}_{j+1,j} = \sum_{k=1}^N E_{(k)}^{j+1,j} \otimes_{\gamma \neq k} G_{(\gamma)}^{j+1,j}(k, \gamma), \quad j = 1, \dots, m+n-1. \quad (5.3)$$

Here the diagonal matrices $G_{(j)}^{\gamma, \gamma \pm 1}(i, j)$ are:

- For $1 < \gamma + 1 \leq m$,

$$(G_{(j)}^{\gamma, \gamma+1}(i, j))_{kl} = \delta_{kl} \begin{cases} 2e^{-\eta} \cosh \eta, & k = \gamma, \\ (2a_{ij} \cosh \eta)^{-1} e^\eta, & k = \gamma + 1, \\ 1, & \text{otherwise,} \end{cases} \quad (5.4)$$

$$(G_{(j)}^{\gamma+1,\gamma}(i,j))_{kl} = \delta_{kl} \begin{cases} 2e^{-\eta} \cosh \eta, & k = \gamma + 1, \\ (2a_{ji} \cosh \eta)^{-1} e^{\eta}, & k = \gamma, \\ 1, & \text{otherwise,} \end{cases} \quad (5.5)$$

- For $\gamma = m$,

$$(G_{(j)}^{\gamma,\gamma+1}(i,j))_{kl} = \delta_{kl} \begin{cases} 2e^{-\eta} \cosh \eta, & k = \gamma, \\ e^{-\eta}, & k = \gamma + 1, \\ 1, & \text{otherwise,} \end{cases} \quad (5.6)$$

$$(G_{(j)}^{\gamma+1,\gamma}(i,j))_{kl} = \delta_{kl} \begin{cases} (2a_{ji} \cosh \eta)^{-1} e^{\eta}, & k = \gamma, \\ (a_{ji})^{-1} e^{\eta}, & k = \gamma + 1, \\ 1, & \text{otherwise,} \end{cases} \quad (5.7)$$

- For $1 + m \leq \gamma < m + n$,

$$(G_{(j)}^{\gamma,\gamma+1}(i,j))_{kl} = \delta_{kl} \begin{cases} (a_{ij})^{-1} e^{\eta}, & k = \gamma, \\ e^{-\eta}, & k = \gamma + 1, \\ 1, & \text{otherwise,} \end{cases} \quad (5.8)$$

$$(G_{(j)}^{\gamma+1,\gamma}(i,j))_{kl} = \delta_{kl} \begin{cases} (a_{ji})^{-1} e^{-\eta}, & k = \gamma + 1, \\ e^{\eta}, & k = \gamma, \\ 1, & \text{otherwise.} \end{cases} \quad (5.9)$$

The proof for this theorem can be found in ^{38,40}. This theorem plays an important role in the construction of the symmetric representations of the creation (annihilation) operators.

Among the matrix elements of the monodromy matrix $T_{i,j}(\lambda)$, the operators $T_{m+n,m+n-l}(\lambda)$ and $T_{m+n-l,m+n}(\lambda)$ ($l = 1, \dots, m+n-1$) are called creation and annihilation operators, respectively, and are usually denoted by

$$C_{m+n-l}(\lambda) = T_{m+n,m+n-l}(\lambda), \quad B_{m+n-l}(\lambda) = T_{m+n-l,m+n}(\lambda). \quad (5.10)$$

In the F -basis, they become

$$\tilde{C}_{m+n-l}(\lambda) = F_{1\dots N} C_{m+n-l}(\lambda) F_{1\dots N}^{-1}, \quad \tilde{B}_{m+n-l}(\lambda) = F_{1\dots N} B_{m+n-l}(\lambda) F_{1\dots N}^{-1}. \quad (5.11)$$

Let us denote $T_{m+n,m+n}(\lambda)$ by $D(\lambda)$ and the corresponding operator in the F -basis by $\tilde{D}(\lambda) = F_{1\dots N} D(\lambda) F_{1\dots N}^{-1}$.

Proposition 3. $\tilde{D}(\lambda)$ is a diagonal matrix given by

$$\tilde{D}(\lambda) = \otimes_{i=1}^N \text{diag}(a_{0i}, \dots, a_{0i}, 1)_{(i)}. \quad (5.12)$$

This can be proved as follows ⁴⁰. From (3.1), we derive that

$$D(\lambda) P_0^{m+n} = T_{m+n,m+n}(\lambda) e_{m+n,m+n} = P_0^{m+n} T_{0,1\dots N}(\lambda) P_0^{m+n}. \quad (5.13)$$

Acting the F -matrix from the left on the both sides of the above equation, we have

$$\begin{aligned} F_{1\dots N}D(\lambda)P_0^{m+n} &= \sum_{\sigma \in S_N} \sum_{\alpha_{\sigma(1)} \dots \alpha_{\sigma(N)}}^* S(\sigma, \alpha_\sigma) \prod_{i=1}^N P_{\sigma(i)}^{\alpha_\sigma} R_{1\dots N}^\sigma P_0^{m+n} T_{0,1\dots N}(\lambda) P_0^{m+n} \\ &= \sum_{\sigma \in S_N} \sum_{\alpha_{\sigma(1)} \dots \alpha_{\sigma(N)}}^* S(\sigma, \alpha_\sigma) \prod_{i=1}^N P_{\sigma(i)}^{\alpha_\sigma} P_0^{m+n} T_{0,\sigma(1\dots N)}(\lambda) P_0^{m+n} R_{1\dots N}^\sigma. \end{aligned} \quad (5.14)$$

Following ³³, we can split the sum \sum^* according to the number of occurrences of the index $m+n$.

$$\begin{aligned} F_{1\dots N}D(\lambda)P_0^{m+n} &= \sum_{\sigma \in S_N} \sum_{k=0}^N \sum_{\alpha_{\sigma(1)} \dots \alpha_{\sigma(N)}}^* S(\sigma, \alpha_\sigma) \prod_{j=N-k+1}^N \delta_{\alpha_{\sigma(j)}, m+n} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \\ &\quad \times \prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} P_0^{m+n} T_{0,\sigma(1\dots N)}(\lambda) P_0^{m+n} R_{1\dots N}^\sigma. \end{aligned} \quad (5.15)$$

Consider the prefactor of $R_{1\dots N}^\sigma$. We have

$$\begin{aligned} &\prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \prod_{j=N-k+1}^N P_{\sigma(j)}^{m+n} P_0^{m+n} T_{0,\sigma(1\dots N)}(\lambda) P_0^{m+n} \\ &= \prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \prod_{j=N-k+1}^N (R_{0\sigma(j)})_{m+n}^{m+n} P_0^{m+n} T_{0,\sigma(1\dots N-k)}(\lambda) P_0^{m+n} \prod_{j=N-k+1}^N P_{\sigma(j)}^{m+n} \\ &= \prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} P_0^{m+n} T_{0,\sigma(1\dots N-k)}(\lambda) P_0^{m+n} \prod_{j=N-k+1}^N P_{\sigma(j)}^{m+n} \\ &= \prod_{i=1}^{N-k} (R_{0\sigma(i)})_{m+n\alpha_{\sigma(i)}}^{m+n\alpha_{\sigma(i)}} \prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \prod_{j=N-k+1}^N P_{\sigma(j)}^{m+n} P_0^{m+n} \\ &= \prod_{i=1}^{N-k} a_{0\sigma(i)} \prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \prod_{j=N-k+1}^N P_{\sigma(j)}^{m+n} P_0^{m+n}, \end{aligned} \quad (5.16)$$

where $a_{0i} = a(\lambda, \xi_i)$. Substituting (5.16) into (5.15), we have

$$F_{1\dots N}D(\lambda) = \otimes_{i=1}^N \text{diag}(a_{0i}, \dots, a_{0i}, 1)_{(i)} F_{1\dots N}. \quad (5.17)$$

By means of the expressions of the generators of $U_q(gl(m|n))$ in theorem 3, combining with the Theorem 1 and Proposition 3, we have

Theorem 4. *In the F -basis the creation operators $C_{m+n-l}(\lambda)$ and annihilation operators $B_{m+n-l}(\lambda)$, ($l = 1, \dots, m+n-1$), are given by*

$$\tilde{C}_{m+n-l}(\lambda) = \left(q^{-(-1)^{[m+n]}} \tilde{E}_{m+n-l, m+n} \tilde{D}(\lambda) - \tilde{D}(\lambda) \tilde{E}_{m+n-l, m+n} \right) q^{-\sum_{k=1}^l H_{m+n-k}}$$

$$-\sum_{\alpha=1}^{l-1} (1-q^{-2(-1)^{[m+n-\alpha]}}) \tilde{C}_{m+n-\alpha}(\lambda) \tilde{E}_{m+n-l, m+n-\alpha} q^{-\sum_{k=\alpha+1}^l H_{m+n-k}}. \quad (5.18)$$

$$\begin{aligned} \tilde{B}_{m+n-l}(\lambda) &= (-1)^{[m+n]+[m+n-l]} q^{\sum_{k=1}^l H_{m+n-k}} \\ &\times \left(q^{(-1)^{[m+n]}} \tilde{D}(\lambda) \tilde{E}_{m+n, m+n-l} - \tilde{E}_{m+n, m+n-l} \tilde{D}(\lambda) \right) \\ &- \sum_{\alpha=1}^{l-1} (-1)^{([m+n]+[m+n-l])([m+n-l]+[m+n-l+\alpha])} (1 - q^{2(-1)^{[m+n-l+\alpha]}}) \\ &\times q^{\sum_{k=l-\alpha+1}^l H_{m+n-k}} \tilde{E}_{m+n-l+\alpha, m+n-l} \tilde{B}_{m+n-l+\alpha}(\lambda), \end{aligned} \quad (5.19)$$

respectively.

For some special values of m and n , we have:

- For $m = 2, n = 0$, i.e. the $U_q(gl(2|0))$ case:

$$\tilde{C}_1(\lambda) = - \sum_{i=1}^N b_{0i}^- E_{(i)}^{1,2} \otimes_{j \neq i} \text{diag} (2a_{0j} \cosh \eta, c_{0j} (2a_{ij} \cosh \eta)^{-1})_{(j)}. \quad (5.20)$$

$$\tilde{B}_1(\lambda) = - \sum_{i=1}^N b_{0i}^+ E_{(i)}^{2,1} \otimes_{j \neq i} \text{diag} (a_{0j} (2a_{ji} \cosh \eta)^{-1}, 2c_{0j} \cosh \eta)_{(j)}, \quad (5.21)$$

- For $m = 2, n = 1$, i.e. the $U_q(gl(2|1))$ case,

$$\tilde{C}_2(\lambda) = \sum_{i=1}^N b_{0i}^- E_{(i)}^{2,3} \otimes_{j \neq i} \text{diag} (a_{0j}, 2a_{0j} \cosh \eta, 1)_{(j)}, \quad (5.22)$$

$$\tilde{B}_2(\lambda) = - \sum_{i=1}^N b_{0i}^+ E_{(i)}^{3,2} \otimes_{j \neq i} \text{diag} (a_{0j}, a_{0j} (2a_{ji} \cosh \eta)^{-1}, a_{ji}^{-1})_{(j)}, \quad (5.23)$$

$$\begin{aligned} \tilde{C}_1(\lambda) &= \sum_{i=1}^N b_{0i}^- E_{(i)}^{1,3} \otimes_{j \neq i} \text{diag} (2a_{0j} \cosh \eta, a_{0j} (a_{ij})^{-1}, 1)_{(j)} \\ &+ \sum_{i \neq j=1}^N \frac{a_{0i} b_{0j}^- (2a_{ij} \sinh \eta + b_{ij}^-)}{a_{ij}} E_{(i)}^{1,2} \otimes E_{(j)}^{2,3} \\ &\otimes_{k \neq i, j} (2a_{0k} \cosh \eta, a_{0k} (a_{ik})^{-1}, 1)_{(k)}. \end{aligned} \quad (5.24)$$

$$\tilde{B}_1(\lambda) = - \sum_{i=1}^N b_{0i}^+ E_{(i)}^{3,1} \otimes_{j \neq i} \text{diag} (a_{0j} (2a_{ji} \cosh \eta)^{-1}, a_{0j} (a_{ji})^{-1}, a_{ji}^{-1})_{(j)}$$

16 *Shao-You Zhao et al*

$$\begin{aligned}
& + \sum_{i \neq j=1}^N \frac{a_{0j} b_{0i}^+ (2a_{ji} \sinh \eta - b_{ji}^+)}{a_{ji}} E_{(i)}^{3,2} \otimes E_{(j)}^{2,1} \\
& \otimes_{k \neq i,j} (a_{0k} (2a_{kj} \cosh \eta)^{-1}, a_{0k} (a_{ki})^{-1}, a_{ki})_{(k)}.
\end{aligned} \tag{5.25}$$

Here $x_{0k} \equiv x(\lambda, \xi)$ with x_{ij} stands for a_{ij}, b_{ij}^\pm .

Thus, one sees that working in the F -basis, the creation and annihilation operators take completely symmetric forms, e.g. (5.20)-(5.25).

6. The q -deformed supersymmetry t - J model

In this section, we will construct the determinant representation of the correlation functions for the q -deformed supersymmetric t - J model, generalizing the results in ³⁷ for the supersymmetric t - J model.

6.1. Bethe states

In the framework of the algebraic ansatz, the Bethe states (eigenstates) are constructed by acting the creation operators $C_i(\lambda)$ (or the annihilation operators $B_i(\lambda)$) on the pseudo-vacuum state.

Definition 3. Let $|0\rangle$ be the pseudo-vacuum state of the N -fold quantum tensor space $V^{\otimes N}$, and $|0\rangle^{(1)}$ be the pseudo-nested-vacuum state of the n_1 -fold nested quantum tensor space $(V^{(1)})^{\otimes n_1}$, i.e.,

$$|0\rangle = \otimes_{i=1}^N \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{(i)}, \quad |0\rangle^{(1)} = \otimes_{j=1}^{n_1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{(j)}. \tag{6.1}$$

The Bethe state of the q -deformed supersymmetric t - J model is then defined by

$$|\Omega_N(\{\lambda_j\})\rangle = \sum_{d_1 \dots d_{n_1}} (\Omega_{n_1}^{(1)})^{d_1 \dots d_{n_1}} C_{d_1}(\lambda_1) \dots C_{d_{n_1}}(\lambda_{n_1}) |0\rangle \quad (\lambda_1 \neq \dots \neq \lambda_{n_1}), \tag{6.2}$$

where $d_i = 1, 2$, $(\Omega_{n_1}^{(1)})^{d_1 \dots d_{n_1}}$ is a component of the nested Bethe state $|\Omega\rangle^{(1)}$ via

$$|\Omega_{n_1}(\{\lambda_j^{(1)}\})\rangle^{(1)} = C_1^{(1)}(\lambda_1^{(1)}) \dots C_1^{(1)}(\lambda_{n_2}^{(1)}) |0\rangle^{(1)} \quad (\lambda_1^{(1)} \neq \dots \neq \lambda_{n_2}^{(1)}), \tag{6.3}$$

and $C^{(1)}$, the creation operator of the nested $U_q(gl(2))$ system, is the lower-triangular entry of the nested monodromy matrix $T^{(1)}$

$$\begin{aligned}
T^{(1)}(\lambda^{(1)}) &= r_{0n_1}(\lambda^{(1)} - \lambda_{n_1}) r_{0n_1-1}(\lambda^{(1)} - \lambda_{n_1-1}) \dots r_{01}(\lambda^{(1)} - \lambda_1) \\
&\equiv \begin{pmatrix} A^{(1)}(\lambda^{(1)}) & B^{(1)}(\lambda^{(1)}) \\ C^{(1)}(\lambda^{(1)}) & D^{(1)}(\lambda^{(1)}) \end{pmatrix}_{(0)}
\end{aligned} \tag{6.4}$$

with $r_{12}(\lambda_1, \lambda_2) \equiv r_{12}(\lambda_1 - \lambda_2) = c_{12}(e_{1,1} \otimes e_{1,1} + e_{2,2} \otimes e_{2,2}) + a_{12}(e_{1,1} \otimes e_{2,2} + e_{2,2} \otimes e_{1,1} - b_{12}^+ e_{2,1} \otimes e_{1,2} - b_{12}^- e_{1,2} \otimes e_{2,1})$.

Similarly, we can also define the dual Bethe state $\langle \Omega_N |$.

Definition 4. With the help of the dual pseudo-vacuum state $\langle 0 |$ and the dual pseudo-nested-vacuum state $\langle 0 |^{(1)}$, the dual Bethe state is defined by

$$\langle \Omega_N(\{\mu_j\}) | = \sum_{f_{n_1}, \dots, f_1} (\Omega^{(1)})^{f_{n_1} \dots f_1} \langle 0 | B_{f_{n_1}}(\mu_{n_1}) \dots B_{f_1}(\mu_1) (\mu_{n_1} \neq \dots \neq \mu_1) \quad (6.5)$$

where $(\Omega^{(1)})^{f_{n_1} \dots f_1}$ is a component of the dual nested Bethe state $\langle \Omega |^{(1)}$

$$\langle \Omega_{n_1}(\{\mu_j^{(1)}\}) |^{(1)} = \langle 0 |^{(1)} B_1^{(1)}(\mu_{n_2}^{(1)}) \dots B_1^{(1)}(\mu_1^{(1)}) \quad (\mu_{n_2}^{(1)} \neq \dots \neq \mu_1^{(1)}). \quad (6.6)$$

The diagonalization of the transfer matrix $t(\lambda)$ leads to the following theorem 50.

Theorem 5. The Bethe states $|\Omega_N(\{\lambda_j\})\rangle$ defined by (6.2) are eigenstates of the transfer matrix $t(\lambda)$ if the spectral parameters λ_j ($j = 1, \dots, n_1$) satisfy the Bethe ansatz equations (BAE)

$$\prod_{k=1}^N a(\lambda_j, \xi_k) \prod_{l=1}^{n_2} a^{-1}(\lambda_j, \lambda_l^{(1)}) = 1 \quad (j = 1, \dots, n_1) \quad (6.7)$$

and the nested Bethe ansatz equations (NBAE)

$$\prod_{j=1}^{n_1} a(\lambda_j, \lambda_l^{(1)}) \prod_{k=1, k \neq l}^{n_2} \frac{a(\lambda_l^{(1)}, \lambda_k^{(1)})}{a(\lambda_k^{(1)}, \lambda_l^{(1)})} = 1 \quad (l = 1, \dots, n_2). \quad (6.8)$$

The eigenvalues $\Lambda(\lambda, \{\lambda_k\}, \{\lambda_j^{(1)}\})$ of the transfer matrix $t(\lambda)$ are given by

$$\begin{aligned} & \Lambda(\lambda, \{\lambda_k\}, \{\lambda_j^{(1)}\}) \\ &= \prod_{i=1}^N a(\lambda, \xi_i) \prod_{j=1}^{n_1} \frac{1}{a(\lambda, \lambda_j)} \Lambda^{(1)}(\lambda) + \prod_{j=1}^{n_1} \frac{1}{a(\lambda_j, \lambda)}, \end{aligned} \quad (6.9)$$

where $\Lambda^{(1)}(\lambda)$ is the eigenvalues of the nested transfer matrix $t^{(1)}(\lambda) = \text{str}_0 T^{(1)}(\lambda)$

$$\Lambda^{(1)}(\lambda) = - \prod_{j=1}^{n_1} \frac{a(\lambda, \lambda_j)}{a(\lambda_j, \lambda)} \prod_{k=1}^{n_2} \frac{1}{a(\lambda, \lambda_k^{(1)})} - \prod_{j=1}^{n_1} a(\lambda, \lambda_j) \prod_{k=1}^{n_2} \frac{1}{a(\lambda^{(1)}, \lambda)}. \quad (6.10)$$

One easily checks that this theorem also holds for the dual Bethe state $\langle \Omega_N(\{\mu_j\}) |$ defined by (6.5) if we change the spectral parameters λ_j and $\lambda_j^{(1)}$ in (6.7)-(6.10) to μ_j and $\mu_j^{(1)}$, respectively.

Let $\sigma = \sigma_{i_1} \dots \sigma_{i_k}$ be any element of the permutation group \mathcal{S}_{n_1} with σ_{i_j} be elementary permutations $\sigma_{i_j}(i_j, i_j + 1) = (i_j + 1, i_j)$. We define the following exchange operator $\hat{f}_\sigma = \hat{f}_{\sigma_{i_1}} \dots \hat{f}_{\sigma_{i_k}}$ for the Bethe state $|\Omega_N(\{\lambda_j\})\rangle$ of the q -deformed supersymmetric t - J model, $\hat{f}_\sigma |\Omega_N(\{\lambda_j\})\rangle = |\Omega_N(\{\lambda_{\sigma(j)}\})\rangle$. Then one may prove^{38,40}, under the action of the exchange operator, we have the following proposition:

Proposition 4. *The Bethe state $|\Omega_N(\{\lambda_j\})\rangle$ of the q -deformed supersymmetric t - J model satisfies the following exchange symmetry*

$$\hat{f}_\sigma |\Omega_N(\{\lambda_j\})\rangle = \frac{1}{c_{1\dots\alpha}^\sigma} |\Omega_N(\{\lambda_j\})\rangle, \quad (6.11)$$

where $c_{1\dots n_1}^\sigma$ has the decomposition law $c_{1\dots n_1}^{\sigma'\sigma} = c_{\sigma'(1\dots n_1)}^\sigma c_{1\dots n_1}^{\sigma'}$, and $c_{1\dots n_1}^{\sigma_i} = c_{i\ i+1} \equiv c(\lambda_i, \lambda_{i+1})$ for an elementary permutation σ_i .

For the non-super case, the element c of the R -matrix (2.7) tends to 1 so the relation (6.11) of the exchange symmetry changes to $\hat{f}_\sigma |\Phi_N(\{\lambda_j\})\rangle = |\Phi_N(\{\lambda_j\})\rangle$, where $|\Phi\rangle$ is the Bethe state of any integrable $U_q(gl(n))$ system. The exchange symmetry for the non-super case was first proposed by ^{51,52}.

6.2. Symmetric representations of the Bethe state

Acting the associated F -matrix on the pseudo-vacuum state $|0\rangle$, one finds that the pseudo-vacuum state is invariant. It is due to the fact that only the term with all roots equal to 3 will produce non-zero results. Therefore, the Bethe state (6.2) in the F -basis $|\tilde{\Omega}_N(\{\lambda_j\})\rangle \equiv F_{1\dots N} |\Omega_N(\{\lambda_j\})\rangle$ can be written as

$$|\tilde{\Omega}_N(\{\lambda_j\})\rangle = \sum_{d_1\dots d_{n_1}} (\Omega_{n_1}^{(1)})^{d_1\dots d_{n_1}} \tilde{C}_{d_1}(\lambda_1) \dots \tilde{C}_{d_{n_1}}(\lambda_{n_1}) |0\rangle. \quad (6.12)$$

Without loss of generality, we will only concentrate on the Bethe state with the **quantum number** p which indicates the number of $d_i = 2$, and will use the notation $|\tilde{\Omega}_N(\{\lambda_j\}_{(p,n_1)})\rangle$ with the subscript pair (p, n_1) to denote a Bethe state which has quantum number p and has n spectral parameters.

Proposition 5. *The Bethe state of the q -deformed supersymmetric t - J model can be represented in the F -basis by*

$$\begin{aligned} |\tilde{\Omega}_N(\{\lambda_j\}_{(p,n_1)})\rangle &= \sum_{\sigma \in S_N} Y_R(\{\lambda_{\sigma(i)}\}, \{\lambda_{\sigma(j)}^{(1)}\}) \\ &\quad \times \tilde{C}_2(\lambda_{\sigma(1)}) \dots \tilde{C}_2(\lambda_{\sigma(p)}) \tilde{C}_1(\lambda_{\sigma(p+1)}) \dots \tilde{C}_1(\lambda_{\sigma(n_1)}) |0\rangle \\ &= \sum_{\sigma \in S_N} Y_R(\{\lambda_{\sigma(i)}\}, \{\lambda_{\sigma(j)}^{(1)}\}) \sum_{i_1 < \dots < i_p} \sum_{i_{p+1} < \dots < i_{n_1}} (2 \cosh \eta)^{\frac{p(p-1) + (n_1-p)(n_1-p-1)}{2}} \end{aligned} \quad (6.13)$$

$$\begin{aligned} &\times \prod_{l=1}^p \prod_{k=p+1}^{n_1} a(\lambda_{\sigma(l)}, \xi_{i_k}) \det \mathcal{B}_p^-(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(p)}; \xi_{i_1}, \dots, \xi_{i_p}) \\ &\times \det \mathcal{B}_{n-p}^-(\lambda_{\sigma(p+1)}, \dots, \lambda_{\sigma(n_1)}; \xi_{i_{p+1}}, \dots, \xi_{i_{n_1}}) \prod_{j=1}^{\overrightarrow{p}} E_{(i_j)}^{23} \prod_{j=p+1}^{\overrightarrow{n_1}} E_{(i_j)}^{13} |0\rangle \end{aligned} \quad (6.14)$$

with the sets $\{i_1, \dots, i_p\} \cap \{i_{p+1}, \dots, i_{n_1}\} = \emptyset$ and the prefactor Y_R being

$$Y_R(\{\lambda_{\sigma(i)}\}, \{\lambda_{\sigma(j)}^{(1)}\}) = \frac{1}{p!(n_1-p)!} c_{1\dots n_1}^\sigma \prod_{k=p+1}^{n_1} \prod_{l=1}^p \left(-\frac{2a(\lambda_{\sigma(k)}, \lambda_{\sigma(l)}) \cosh \eta}{a(\lambda_{\sigma(l)}, \lambda_{\sigma(k)})} \right)$$

$$\times B_{n-p}^* \left(\lambda_{p+1}^{(1)}, \dots, \lambda_{n_1}^{(1)} | \lambda_{\sigma(p+1)}, \dots, \lambda_{\sigma(n_1)} \right). \quad (6.15)$$

Here c_{1,\dots,n_1}^σ has the decomposition law $c_{1\dots n_1}^{\sigma'\sigma} = c_{\sigma'(1\dots n_1)}^\sigma c_{1\dots n_1}^{\sigma'}$ with $c_{1\dots n_1}^{\sigma_i} = c_{i+1} \equiv c(\lambda_i, \lambda_{i+1})$ for an elementary permutation σ_i , the c -number B_p^* is given by

$$B_p^* \left(\lambda_1^{(1)}, \dots, \lambda_p^{(1)} | \lambda_1, \dots, \lambda_p \right) = \sum_{\sigma \in \mathcal{S}_p} \prod_{k=1}^p \left(-b^-(\lambda_k^{(1)}, \lambda_{\sigma(k)}) \right) \\ \times \prod_{j \neq \sigma(k), \dots, \sigma(p)} \frac{c(\lambda_k^{(1)}, \lambda_j)}{2a(\lambda_{\sigma(k)}, \lambda_j) \cosh \eta} \prod_{l=k+1}^p 2a(\lambda_k^{(1)}, \lambda_{\sigma(l)}) \cosh \eta,$$

and the elements of the $n_1 \times n_1$ matrix $\mathcal{B}_{n_1}^\pm(\{\lambda_i\}; \{\xi_j\})$ are

$$(\mathcal{B}_{n_1}^\pm)_{\alpha\beta} = b^\pm(\lambda_\alpha, \xi_\beta) \prod_{\gamma=1}^{\alpha-1} a(\lambda_\gamma, \xi_\beta). \quad (6.16)$$

In (6.14), we have used the convention $\prod_{i=1}^{\overrightarrow{n_1}} f_i \equiv f_1 \dots f_{n_1}$. For our later use, we also

introduce the notation $\prod_{i=1}^{\overleftarrow{n_1}} f_i \equiv f_{n_1} \dots f_1$.

Proof. The proof is similar to that for the supersymmetric t - J model⁴⁰. Define $|\tilde{\Omega}_{n_1}^{(1)}(\{\lambda_j^{(1)}\})\rangle \equiv F_{1\dots n_1}^{(1)} |\Omega_{n_1}^{(1)}(\{\lambda_j^{(1)}\})\rangle$, where $F_{1\dots n_1}^{(1)}$ is the F -matrix (4.2) with $m = 2, n = 0$. Then substituting the expression \tilde{C}_1 (5.20) into the nested Bethe state (6.3), we have

$$|\tilde{\Omega}_{n_1}^{(1)}(\{\lambda_j^{(1)}\})\rangle = s(c) \tilde{C}^{(1)}(\lambda_1^{(1)}) \dots \tilde{C}^{(1)}(\lambda_{n_2}^{(1)}) |0\rangle^{(1)} \\ = s(c) \sum_{i_1 < \dots < i_{n_2}} B_{n_2}^{(1)}(\lambda_1^{(1)}, \dots, \lambda_{n_2}^{(1)} | \lambda_{i_1}, \dots, \lambda_{i_{n_2}}) E_{(i_1)}^{1,2} \dots E_{(i_{n_2})}^{1,2} |0\rangle^{(1)}, \quad (6.17)$$

where $s(c) = \prod_{i < j} (1 + c_{ij})$, and

$$B_{n_2}^{(1)}(\lambda_1^{(1)}, \dots, \lambda_{n_2}^{(1)} | \lambda_1, \dots, \lambda_{n_2}) = \sum_{\sigma \in \mathcal{S}_{n_2}} \prod_{k=1}^{n_2} \left(-b^-(\lambda_k^{(1)}, \lambda_{\sigma(k)}) \right) \\ \times \prod_{j \neq \sigma(k), \dots, \sigma(n_2)} \frac{c(\lambda_k^{(1)}, \lambda_j)}{2a(\lambda_{\sigma(k)}, \lambda_j) \cosh \eta} \prod_{l=k+1}^{n_2} 2a(\lambda_k^{(1)}, \lambda_{\sigma(l)}) \cosh \eta. \quad (6.18)$$

Now we study the Bethe state (6.2) of the quantum supersymmetric t - J model.

$$|\tilde{\Omega}_N(\{\lambda_j\})\rangle = \sum_{d_1 \dots d_{n_1}} (\Omega_{n_1}^{(1)})^{d_1 \dots d_{n_1}} \tilde{C}_{d_1}(\lambda_1) \dots \tilde{C}_{d_{n_1}}(\lambda_{n_1}) |0\rangle \\ = (\Omega_{n_1}^{(1)})^{1\dots 12\dots 2} \tilde{C}_1(\lambda_{p+1}) \dots \tilde{C}_1(\lambda_{n_1}) \tilde{C}_2(\lambda_1) \dots \tilde{C}_2(\lambda_p) |0\rangle + \text{other terms}$$

20 Shao-You Zhao et al

$$\begin{aligned}
&= \frac{1}{p!(n_1 - p)!} \sum_{\sigma \in \mathcal{S}_{n_1}} c_{1\dots n_1}^\sigma (\Omega_{n_1}^{(1),\sigma})^{11\dots 12\dots 2} \prod_{k=p+1}^{n_1} \prod_{l=1}^p \left(-\frac{1}{a(\lambda_{\sigma(l)}, \lambda_{\sigma(k)})} \right) \\
&\quad \times \tilde{C}_2(\lambda_{\sigma(1)}) \dots \tilde{C}_2(\lambda_{\sigma(p)}) \tilde{C}_1(\lambda_{\sigma(p+1)}) \dots \tilde{C}_1(\lambda_{\sigma(n_1)}) |0\rangle, \tag{6.19}
\end{aligned}$$

where we have used the proposition 4 and the commutation relation

$$\tilde{C}_i(\mu) \tilde{C}_j(\lambda) = -\frac{1}{a(\lambda, \mu)} \tilde{C}_j(\lambda) \tilde{C}_i(\mu) + \frac{b(\lambda, \mu)}{a(\lambda, \mu)} \tilde{C}_j(\mu) \tilde{C}_i(\lambda). \tag{6.20}$$

Considering the $1\dots 12\dots 2$ component of the nested Bethe state $|\tilde{\Omega}_{n_1}^{(1)}(\{\lambda_j^{(1)}\})\rangle$, one easily proves⁴⁰ the relation $(\tilde{\Omega}_{n_1}^{(1)})^{11\dots 12\dots 2} = t(c)(\Omega_{n_1}^{(1)})^{11\dots 12\dots 2}$ with the scalar factor $t(c) = \prod_{j>i=1}^p (1 + \bar{c}_{ij}) \prod_{j>i=p+1}^{n_1} (1 + \bar{c}_{ij})$, and $\bar{c}_{ij} = c(\lambda_i, \lambda_j)$. Then substituting the expressions \tilde{C}_1 (5.24) and \tilde{C}_2 (5.22) into (6.19), one obtains (6.14). Therefore we have proved the proposition 4. \square

By a similar procedure, one may prove the following proposition for the dual Bethe state $\langle \tilde{\Omega}_N(\{\mu_j\}_{(p, n_1)}) |$ (6.5):

Proposition 6. *The dual Bethe state $\langle \tilde{\Omega}_N(\{\mu_j\}_{(p, n_1)}) |$ of the $U_q(gl(2|1))$ supersymmetric t - J model can be represented by*

$$\begin{aligned}
&\langle \tilde{\Omega}_N(\{\mu_j\}_{(p, n_1)}) | = \sum_{\sigma \in \mathcal{S}_N} Y_L(\{\mu_{\sigma(i)}\}, \{\mu_{\sigma(j)}^{(1)}\}) \langle 0 | \tilde{B}_1(\mu_{\sigma(n_1)}) \dots \tilde{B}_1(\mu_{\sigma(p+1)}) \\
&\quad \times \tilde{B}_2(\mu_{\sigma(p)}) \dots \tilde{B}_2(\mu_{\sigma(1)}) \tag{6.21} \\
&= \sum_{\sigma \in \mathcal{S}_N} Y_L(\{\mu_{\sigma(i)}\}, \{\mu_{\sigma(j)}^{(1)}\}) \sum_{i_1 < \dots < i_p} \sum_{i_{p+1} < \dots < i_{n_1}} (-1)^{n_1} (2 \cosh \eta)^{-\frac{p(p-1) + (n_1-p)(n_1-p-1)}{2}} \\
&\quad \times \prod_{l=1}^p \prod_{k=p+1}^{n_1} a(\mu_{\sigma(l)}, \xi_{i_k}) \prod_{l=1}^p \prod_{k \neq i_l, i_{p+1}, \dots, i_{n_1}}^N a^{-1}(\xi_k, \xi_{i_l}) \prod_{l=p+1}^{n_1} \prod_{k=1, \neq i_l}^N a^{-1}(\xi_k, \xi_{i_l}) \\
&\quad \times \det \mathcal{B}_p^+(\mu_{\sigma(1)}, \dots, \mu_{\sigma(p)}; \xi_{i_1}, \dots, \xi_{i_p}) \\
&\quad \times \det \mathcal{B}_{n_1-p}^+(\mu_{\sigma(p+1)}, \dots, \mu_{\sigma(n_1)}; \xi_{i_{p+1}}, \dots, \xi_{i_{n_1}}) \langle 0 | \overleftarrow{\prod_{j=p+1}^{n_1}} E_{(i_j)}^{31} \overleftarrow{\prod_{j=1}^p} E_{(i_j)}^{32}, \tag{6.22}
\end{aligned}$$

where the prefactor Y_L is

$$\begin{aligned}
&Y_L(\{\mu_{\sigma(i)}\}, \{\mu_{\sigma(j)}^{(1)}\}) \\
&= \frac{1}{p!(n_1 - p)!} c_{1\dots n_1}^\sigma B_{n-p}^{**} \left(\mu_{p+1}^{(1)}, \dots, \mu_{n_1}^{(1)} | \mu_{\sigma(p+1)}, \dots, \mu_{\sigma(n_1)} \right) \\
&\quad \times \prod_{k=p+1}^{n_1} \prod_{l=1}^p \left(-\frac{2a(\mu_{\sigma(k)}, \mu_{\sigma(l)}) \cosh \eta}{a(\mu_{\sigma(l)}, \mu_{\sigma(k)})} \right), \tag{6.23}
\end{aligned}$$

and the c -number B_p^{**} is given by

$$B_p^{**}(\mu_1^{(1)}, \dots, \mu_p^{(1)} | \mu_1, \dots, \mu_p) = \sum_{\sigma \in \mathcal{S}_p} \prod_{k=1}^p b^+(\mu_k^{(1)}, \mu_{\sigma(k)}) \prod_{j \neq \sigma(k), \dots, \sigma(p)} c(\mu_k^{(1)}, \mu_j) \prod_{l=k+1}^p \frac{a(\mu_k^{(1)}, \mu_{\sigma(l)})}{2a(\mu_{\sigma(l)}, \mu_{\sigma(k)}) \cosh \eta}.$$

6.3. Determinant representation of the scalar product

The scalar product of the Bethe states with a given quantum number p is defined by

$$\mathbb{P}_{n_1}(\{\mu_i\}_{(p, n_1)}, \{\lambda_j\}_{(p, n_1)}) = \langle \Omega_N(\{\mu_j\}_{(p, n_1)}) | \Omega_N(\{\lambda_j\}_{(p, n_1)}) \rangle. \quad (6.24)$$

The invariant property of the pseudo-vacuum state under the F -transformation, i.e. $F_{1\dots N}|0\rangle = |0\rangle$ and $\langle 0|F_{1\dots N}^{-1} = \langle 0|$, implies that in the F -basis, the scalar product \mathbb{P}_{n_1} is

$$\begin{aligned} \mathbb{P}_{n_1}(\{\mu_i\}_{(p, n_1)}, \{\lambda_j\}_{(p, n_1)}) &= \langle \tilde{\Omega}_N(\{\mu_j\}_{(p, n_1)}) | \tilde{\Omega}_N(\{\lambda_j\}_{(p, n_1)}) \rangle \\ &= \sum_{\sigma', \sigma} Y_L(\{\mu_{\sigma'(j)}\}, \{\mu_{\sigma'(k)}^{(1)}\}) Y_R(\{\lambda_{\sigma(j)}\}, \{\lambda_{\sigma(k)}^{(1)}\}) \\ &\quad \times \langle 0 | \tilde{B}_1(\mu_{\sigma'(n_1)}) \dots \tilde{B}_1(\mu_{\sigma'(p+1)}) \tilde{B}_2(\mu_{\sigma'(p)}) \dots \tilde{B}_2(\mu_{\sigma'(1)}) \\ &\quad \times \tilde{C}_2(\lambda_{\sigma(1)}) \dots \tilde{C}_2(\lambda_{\sigma(p)}) \tilde{C}_1(\lambda_{\sigma(p+1)}) \dots \tilde{C}_1(\lambda_{\sigma(n_1)}) | 0 \rangle. \end{aligned} \quad (6.25)$$

To compute the scalar product, following ^{29,37}, we introduce the following intermediate functions $G^{(m)}$ (in this section, the integer m is the index of the intermediate function)

$$G^{(m)}(\{\lambda_k\}_{(p, n_1)}, \mu_1, \dots, \mu_m, i_{m+1}, \dots, i_{n_1}) = \begin{cases} \langle 0 | \prod_{k=p+1}^{n_1} E_{(i_k)}^{31} \prod_{k=m+1}^p E_{(i_k)}^{32} \tilde{B}_2(\mu_m) \dots \tilde{B}_2(\mu_1) \\ \quad \times \tilde{C}_2(\lambda_1) \dots \tilde{C}_2(\lambda_p) \tilde{C}_1(\lambda_{p+1}) \dots \tilde{C}_1(\lambda_{n_1}) | 0 \rangle & \text{for } m \leq p, \\ \langle 0 | \prod_{k=m+1}^{n_1} E_{(i_k)}^{31} \tilde{B}_1(\mu_m) \dots \tilde{B}_1(\mu_{p+1}) \tilde{B}_2(\mu_p) \tilde{B}_2(\mu_1) \\ \quad \times \tilde{C}_2(\lambda_1) \dots \tilde{C}_2(\lambda_p) \tilde{C}_1(\lambda_{p+1}) \dots \tilde{C}_1(\lambda_{n_1}) | 0 \rangle & \text{for } m \geq p+1. \end{cases} \quad (6.26)$$

where the lower indices of $E_{(i_k)}^{32}$ and $E_{(i_k)}^{31}$ satisfy the relations $i_{m+1} < \dots < i_p$, $i_{p+1} < \dots < i_{n_1}$ and $\{i_1, \dots, i_p\} \cap \{i_{p+1}, \dots, i_{n_1}\} = \emptyset$. Thus, the scalar product can be rewritten as

$$\begin{aligned} \mathbb{P}_{n_1}(\{\mu_i\}_{(p, n_1)}, \{\lambda_j\}_{(p, n_1)}) &= \sum_{\sigma, \sigma'} Y_L(\{\mu_{\sigma'(j)}\}, \{\mu_{\sigma'(k)}^{(1)}\}) \\ &\quad \times Y_R(\{\lambda_{\sigma(j)}\}, \{\lambda_{\sigma(k)}^{(1)}\}) G^{(n_1)}(\{\lambda_{\sigma(j)}\}_{(p, n_1)}, \{\mu_{\sigma'(k)}\}_{(p, n_1)}). \end{aligned} \quad (6.27)$$

We now compute $G^{(m)}$ for $m \leq p$ and $m \geq p+1$ separately. The procedure is similar to that for the supersymmetric t - J model ³⁷.

22 Shao-You Zhao et al

- $1 \leq m \leq p$

We first compute the function $G^{(m)}$ for $m \leq p$.

Inserting a complete set, (6.26) becomes

$$G^{(m)}(\{\lambda_k\}_{(p,n_1)}, \mu_1, \dots, \mu_m, i_{m+1}, \dots, i_{n_1}) = \sum_{j \neq i_{m+1}, \dots, i_{n_1}}^N \langle 0 | \prod_{k=p+1}^{\overleftarrow{n_1}} E_{(i_k)}^{31} \prod_{k=m+1}^{\overleftarrow{p}} E_{(i_k)}^{32} \tilde{B}_2(\mu_m) \prod_{k=m+1}^{\overrightarrow{m+q}} E_{(i_k)}^{23} E_{(j)}^{23} \prod_{k=m+q+1}^{\overrightarrow{p}} E_{(i_k)}^{23} \prod_{k=p+1}^{\overrightarrow{n_1}} E_{(i_k)}^{13} | 0 \rangle \times G^{(m-1)}(\{\lambda_k\}_{(p,n_1)}, \mu_1, \dots, \mu_{m-1}, i_{m+1}, \dots, i_{m+q}, j, i_{m+q+1}, \dots, i_{n_1}), \quad (6.28)$$

where $(0 \leq q \leq p-m)$. In view of (5.23), we have

$$\begin{aligned} & \langle 0 | \prod_{k=p+1}^{\overleftarrow{n_1}} E_{(i_k)}^{31} \prod_{k=m+1}^{\overleftarrow{p}} E_{(i_k)}^{32} \tilde{B}_2(\mu_m) \prod_{k=m+1}^{\overrightarrow{m+q}} E_{(i_k)}^{23} E_{(j)}^{23} \prod_{k=m+q+1}^{\overrightarrow{p}} E_{(i_k)}^{23} \prod_{k=p+1}^{\overrightarrow{n_1}} E_{(i_k)}^{13} | 0 \rangle \\ &= -(-1)^q (2 \cosh \eta)^{-(p-m)} \cdot b^+(\mu_m, \xi_j) \prod_{l=m+1}^p a(\mu_m, \xi_{i_l}) \\ & \times \prod_{l=p+1}^{n_1} a(\mu_m, \xi_{i_l}) \prod_{k \neq j, i_{p+1}, \dots, i_{n_1}}^N a^{-1}(\xi_k, \xi_j). \end{aligned} \quad (6.29)$$

Substituting the expressions of \tilde{C}_1 (5.24) and \tilde{C}_2 (5.22) into (6.26), we obtain $G^{(0)}$:

$$\begin{aligned} G^{(0)}(\{\lambda_k\}_{(p,n_1)}, i_1, \dots, i_{n_1}) &= \langle 0 | \prod_{k=p+1}^{\overleftarrow{n_1}} E_{(i_k)}^{31} \prod_{k=1}^{\overleftarrow{p}} E_{(i_k)}^{32} \prod_{k=1}^p \tilde{C}_2(\lambda_k) \prod_{k=p+1}^{n_1} \tilde{C}_1(\lambda_k) | 0 \rangle \\ &= 2^{\frac{p(p-1)+(n_1-p)(n_1-p-1)}{2}} \prod_{l=1}^p \prod_{k=p+1}^{n_1} a(\lambda_l, \xi_{i_k}) \det \mathcal{B}_p(\lambda_1, \dots, \lambda_p; \xi_{i_1}, \dots, \xi_{i_p}) \\ & \times \det \mathcal{B}_{n_1-p}(\lambda_{p+1}, \dots, \lambda_{n_1}; \xi_{i_{p+1}}, \dots, \xi_{i_{n_1}}). \end{aligned} \quad (6.30)$$

We compute $G^{(m)}$ by using the recursion relation (6.28). One sees that there are two different determinants in $G^{(0)}$, which are labelled by different λ 's and ξ_{i_k} 's. For $m \leq p$ we only focus on the first determinant, i.e. $\det \mathcal{B}_p$.

To compute $G^{(1)}$, we substitute (6.29) and (6.30) into (6.28) to obtain

$$\begin{aligned} & G^{(1)}(\{\lambda_k\}_{(p,n_1)}, \mu_1, i_2, \dots, i_{n_1}) \\ &= \sum_{j \neq i_2, \dots, i_{n_1}}^N \langle 0 | \prod_{k=p+1}^{\overleftarrow{n_1}} E_{(i_k)}^{31} \prod_{k=2}^{\overleftarrow{p}} E_{(i_k)}^{32} \tilde{B}_2(\mu_1) \prod_{k=2}^{\overrightarrow{q+1}} E_{(i_k)}^{23} E_{(j)}^{23} \prod_{q+2}^{\overrightarrow{p}} E_{(i_k)}^{23} \prod_{k=p+1}^{\overrightarrow{n_1}} E_{(i_k)}^{13} | 0 \rangle \\ & \times G^{(0)}(\{\lambda_k\}_{(p,n_1)}, i_2, \dots, i_{q+1}, j, i_{q+2}, \dots, i_{n_1}) \\ &= -(2 \cosh \eta)^{\frac{(p-1)(p-2)+(n-p)(n-p-1)}{2}} \prod_{l=1}^p \prod_{k=p+1}^{n_1} a(\lambda_l, \xi_{i_k}) \sum_{j \neq i_2, \dots, i_{n_1}}^N (-1)^q b^+(\mu_1, \xi_j) \end{aligned}$$

$$\begin{aligned}
& \times \prod_{l=2}^p a(\mu_1, \xi_{i_l}) \prod_{k \neq j, i_{p+1}, \dots, i_{n_1}}^N a^{-1}(\xi_k, \xi_j) \\
& \times \det \mathcal{B}_p^{-}(\lambda_1, \dots, \lambda_p; \xi_{i_2}, \dots, \xi_{i_{q+1}}, \xi_j, \xi_{i_{q+2}}, \dots, \xi_{i_p}) \\
& \times \prod_{l=p+1}^{n_1} a(\mu_1, \xi_{i_l}) \det \mathcal{B}_{n-p}^{-}(\lambda_{p+1}, \dots, \lambda_{n_1}; \xi_{i_{p+1}}, \dots, \xi_{i_{n_1}}). \tag{6.31}
\end{aligned}$$

Let λ_k ($k = 1, \dots, n$) label the row and ξ_l ($l = i_2, \dots, j, \dots, i_p$) label the column of the matrix \mathcal{B}_p . From (6.30), one sees that the column indices in (6.31) satisfy the sequence $i_2 < \dots < j < \dots < i_p$. Therefore, moving the j th column in the matrix \mathcal{B}_p to the first column, we have

$$\begin{aligned}
& G^{(1)}(\{\lambda_k\}_{(p, n_1)}, \mu_1, i_2, \dots, i_{n_1}) \\
& = -(2 \cosh \eta)^{\frac{(p-1)(p-2)+(n-p)(n-p-1)}{2}} \prod_{l=1}^p \prod_{k=p+1}^{n_1} a(\lambda_l, \xi_{i_k}) \sum_{j \neq i_2, \dots, i_{n_1}}^N b^+(\mu_1, \xi_j) \\
& \times \prod_{l=2}^p a(\mu_1, \xi_{i_l}) \prod_{k \neq j, i_{p+1}, \dots, i_{n_1}}^N a^{-1}(\xi_k, \xi_j) \det \mathcal{B}_p^{-}(\lambda_1, \dots, \lambda_p; \xi_j, \xi_{i_2}, \dots, \xi_{i_{n_1}}) \\
& \times \det \mathcal{B}_{n-p}^{-}(\lambda_{p+1}, \dots, \lambda_{n_1}; \xi_{i_{p+1}}, \dots, \xi_{i_{n_1}}) \\
& = -(2 \cosh \eta)^{\frac{(p-1)(p-2)+(n_1-p)(n_1-p-1)}{2}} \det(\mathcal{B}^{-})_p^{(1)}(\lambda_1, \dots, \lambda_p; \mu_1, \xi_{i_2}, \dots, \xi_{i_p}) \\
& \times \prod_{l=p+1}^{n_1} a(\mu_1, \xi_{i_l}) \det \mathcal{B}_{n_1-p}^{-}(\lambda_{p+1}, \dots, \lambda_{n_1}; \xi_{i_{p+1}}, \dots, \xi_{i_{n_1}}), \tag{6.32}
\end{aligned}$$

where the matrix $(\mathcal{B}^{-})_p^{(1)}(\{\lambda_k\}, \mu_1, \xi_{i_2}, \dots, \xi_{i_p})$ is given by

$$((\mathcal{B}^{-})_p^{(1)})_{\alpha\beta} = \prod_{k=p+1}^{n_1} a(\lambda_\alpha, \xi_{i_k}) a(\mu_1, \xi_{i_\beta}) (\mathcal{B}_p^{-})_{\alpha\beta} \quad (1 \leq \alpha \leq p \text{ and } 2 \leq \beta \leq p), \tag{6.33}$$

$$\begin{aligned}
((\mathcal{B}^{-})_p^{(1)})_{\alpha 1} & = \prod_{k=p+1}^{n_1} a(\lambda_\alpha, \xi_{i_k}) \sum_{j \neq i_2, \dots, i_{n_1}}^N b^+(\mu_1, \xi_j) b^-(\lambda_\alpha, \xi_j) \prod_{\gamma=1}^{\alpha-1} a(\lambda_\gamma, \xi_j) \\
& \times \prod_{k \neq j, i_{p+1}, \dots, i_{n_1}}^N a^{-1}(\xi_k, \xi_j) \quad (1 \leq \alpha \leq p). \tag{6.34}
\end{aligned}$$

Using the properties of determinant, one finds that if $j = i_2, \dots, i_p$, the corresponding terms in (6.34) contribute zero to the determinant. Thus, without changing the determinant of the matrix $\mathcal{B}_p^{(1)}$, the elements $(\mathcal{B}_p^{(1)})_{\alpha 1}$ in (6.34) may be replaced by

$$((\mathcal{B}^{-})_p^{(1)})_{\alpha 1} = e^{\mu_1} f(\mu_1) \equiv e^{\mu_1} \cdot \prod_{k=p+1}^{n_1} a(\lambda_\alpha, \xi_{i_k}) \sum_{j \neq i_{p+1}, \dots, i_{n_1}}^N e^{-\mu_1}$$

24 *Shao-You Zhao et al*

$$\times b^+(\mu_1, \xi_j) b^-(\lambda_\alpha, \xi_j) \prod_{\gamma=1}^{\alpha-1} a(\lambda_\gamma, \xi_j) \prod_{k \neq j, i_{p+1}, \dots, i_{n_1}}^N a^{-1}(\xi_k, \xi_j). \quad (6.35)$$

Thanks to the Bethe ansatz equation (6.7), we may construct the function

$$\begin{aligned} \mathcal{M}_{\alpha\beta}^\pm &= \frac{b^\pm(\lambda_\alpha, \mu_\beta)}{a(\lambda_\alpha, \mu_\beta)} \prod_{\gamma=1}^{\alpha-1} a^{-1}(\lambda_\gamma, \mu_\beta) \prod_{\epsilon=1}^{\beta-1} a^{-1}(\mu_\epsilon, \mu_\beta) \\ &\quad \times \left[\prod_{j=p+1}^{n_1} a(\mu_\beta, \lambda_j^{(1)}) - \prod_{k=1}^N a(\mu_\beta, \xi_k) \prod_{j=p+1}^{n_1} a^{-1}(\mu_\beta, \xi_{i_j}) a(\lambda_\alpha, \xi_{i_j}) \right] \\ &+ \sum_{j=p+1}^{n_1} \left[b^\mp(\mu_\beta, \lambda_j^{(1)}) b^\pm(\lambda_\alpha, \lambda_j^{(1)}) \prod_{\gamma=1}^{\alpha-1} a(\lambda_\gamma, \lambda_j^{(1)}) \right. \\ &\quad \times \left. \prod_{\epsilon=1}^{\beta-1} a(\mu_\epsilon, \lambda_j^{(1)}) \prod_{k=p+1, \neq j}^{n_1} a^{-1}(\lambda_k^{(1)}, \lambda_j^{(1)}) \right] \\ &- \sum_{\gamma=1}^{\alpha-1} \frac{b^\mp(\mu_\beta, \lambda_\gamma)}{a(\mu_\beta, \lambda_\gamma)} \frac{b^\pm(\lambda_\alpha, \lambda_\gamma)}{a(\lambda_\alpha, \lambda_\gamma)} \prod_{\iota=1, \neq \gamma}^{\alpha-1} a^{-1}(\lambda_\gamma, \lambda_\iota) \prod_{\epsilon=1}^{\beta-1} a^{-1}(\lambda_\gamma, \mu_\epsilon) \\ &\quad \times \left[\prod_{j=p+1}^{n_1} a(\lambda_\gamma, \lambda_j^{(1)}) - \prod_{k=1}^N a(\lambda_\gamma, \xi_k) \prod_{j=p+1}^{n_1} a^{-1}(\lambda_\gamma, \xi_{i_j}) a(\lambda_\alpha, \xi_{i_j}) \right] \\ &- \sum_{\epsilon=1}^{\beta-1} \frac{b^\mp(\mu_\beta, \mu_\epsilon)}{a(\mu_\beta, \mu_\epsilon)} \frac{b^\pm(\lambda_\alpha, \mu_\epsilon)}{a(\lambda_\alpha, \mu_\epsilon)} \prod_{\gamma=1}^{\alpha-1} a^{-1}(\mu_\epsilon, \lambda_\gamma) \prod_{\varepsilon=1, \neq \epsilon}^{\beta-1} a^{-1}(\mu_\epsilon, \mu_\varepsilon) \\ &\quad \times \left[\prod_{j=p+1}^{n_1} a(\mu_\epsilon, \lambda_j^{(1)}) - \prod_{k=1}^N a(\mu_\epsilon, \xi_k) \prod_{j=p+1}^{n_1} a^{-1}(\mu_\epsilon, \xi_{i_j}) a(\lambda_\alpha, \xi_{i_j}) \right], \quad (6.36) \end{aligned}$$

where $\lambda_j^{(1)}$ ($j = p+1, \dots, n$) satisfy the NBAE (6.8). By direct computation, one sees that the residues of $e^{-\mu_1} \mathcal{M}_{\alpha 1}^-$ at points $\mu_1 = \lambda_j^{(1)} - \eta \bmod(i\pi)$, $\mu_1 = \lambda_\gamma \bmod(i\pi)$ ($\gamma = 1, \dots, \alpha-1$) and $\mu_1 = \mu_\epsilon \bmod(i\pi)$ ($\epsilon = 1, \dots, \beta-1$) are zero. Moreover the residues of $e^{-\mu_1} \mathcal{M}_{\alpha 1}^-$ at the points $\mu_1 = \lambda_\alpha \bmod(i\pi)$ are also zero because $\lambda_\alpha \bmod(i\pi)$ is a solution of the BAE (6.7). Then comparing (6.35) with (6.36), one finds that as functions of μ_1 , the functions $f(\mu_1)$ and $e^{-\mu_1} \mathcal{M}_{\alpha 1}^-$ have the same residues at the simple poles $\mu_1 = \xi_j - \eta \bmod(i\pi)$ ($j \neq i_{p+1}, \dots, i_{n_1}$), and that when $\mu_1 \rightarrow \infty$, they are bounded. Therefore, according to the properties of the analytic functions (Liouville theorem), we have $((\mathcal{B}^-)_p^{(1)})_{\alpha 1} = \mathcal{M}_{\alpha 1}^-$.

Then, by using the function $G^{(0)}, G^{(1)}$ and the intermediate function (6.28) repeatedly, we obtain $G^{(m)}$ ($m \leq p$):

$$\begin{aligned} &G^{(m)}(\{\lambda_k\}_{(p, n_1)}, \mu_1, \dots, \mu_m, i_{m+1}, \dots, i_{n_1}) \\ &= (-1)^m (2 \cosh \eta)^{\frac{p(p-1)-m(2p-m-1)+(n_1-p)(n_1-p-1)}{2}} \end{aligned}$$

$$\begin{aligned} & \times \det(\mathcal{B}^-)^{(m)}_p(\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_m, i_{m+1}, \dots, i_p) \\ & \times \prod_{l=1}^m \prod_{k=p+1}^{n_1} a(\mu_l, \xi_{i_k}) \det \mathcal{B}^-_{n_1-p}(\lambda_{p+1}, \dots, \lambda_{n_1}; \xi_{i_{p+1}}, \dots, \xi_{i_{n_1}}) \end{aligned} \quad (6.37)$$

with the matrix elements

$$\begin{aligned} ((\mathcal{B}^-)^{(m)}_p)_{\alpha\beta} &= \prod_{\epsilon=1}^{\beta-1} a(\mu_\epsilon, \xi_{i_\beta}) (\mathcal{B}^-_p)_{\alpha\beta}, \quad (1 \leq \alpha \leq p, m < \beta \leq p), \\ ((\mathcal{B}^-)^{(m)}_p)_{\alpha\beta} &= \mathcal{M}^-_{\alpha\beta}, \quad (1 \leq \alpha \leq p, 1 \leq \beta \leq m). \end{aligned} \quad (6.38)$$

(6.37) can be proved by induction. Firstly from (6.32), (6.33) and (6.36), (6.37) is true for $m = 1$. Assume (6.37) for $G^{(m-1)}$. Let us show (6.37) for general m . Substituting $G^{(m-1)}$ and (6.29) into intermediate function (6.28), we have

$$\begin{aligned} & G^{(m)}(\{\lambda_k\}_{(p, n_1)}, \mu_1, \dots, \mu_m, i_{m+1}, \dots, i_{n_1}) = -(-1)^q (2 \cosh \eta)^{-(p-m)} \\ & \times \sum_{j \neq i_{m+1}, \dots, i_{n_1}}^N b^+(\mu_m, \xi_j) \prod_{l=m+1}^{n_1} a(\mu_m, \xi_{i_l}) \prod_{k \neq j, p+1, \dots, n}^N a^{-1}(\xi_k, \xi_j) \\ & \times G^{(m-1)}(\{\lambda_k\}_{(p, n_1)}, \mu_1, \dots, \mu_{m-1}, i_{m+1}, \dots, i_{m+p}, j, i_{m+p+1}, \dots, i_{n_1}) \\ & = -(2 \cosh \eta)^{-(p-m)} \sum_{j \neq i_{m+1}, \dots, i_{n_1}}^N b^+(\mu_m, \xi_j) \prod_{l=m+1}^{n_1} a(\mu_m, \xi_{i_l}) \prod_{k \neq j, p+1, \dots, n_1}^N a^{-1}(\xi_k, \xi_j) \\ & \times G^{(m-1)}(\{\lambda_k\}_{(p, n_1)}, \mu_1, \dots, \mu_{m-1}, j, i_{m+1}, \dots, i_{n_1}) \\ & = (-1)^m (2 \cosh \eta)^{\frac{p(p-1)-m(2p-m-1)+(n_1-p)(n_1-p+1)}{2}} \\ & \times \det(\mathcal{B}^-)^{(m)}_p(\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_m, i_{m+1}, \dots, i_p) \\ & \times \prod_{l=1}^m \prod_{k=p+1}^{n_1} a(\mu_l, \xi_{i_k}) \det \mathcal{B}^-_{n_1-p}(\lambda_{p+1}, \dots, \lambda_{n_1}; \xi_{i_{p+1}}, \dots, \xi_{i_{n_1}}), \end{aligned} \quad (6.39)$$

where the matrix elements $(\mathcal{B}^{(m)}_p)_{\alpha\beta}$ are given by

$$\begin{aligned} ((\mathcal{B}^-)^{(m)}_p)_{\alpha\beta} &= \prod_{\epsilon=1}^{\beta-1} a(\mu_\epsilon, \xi_{i_\beta}) \prod_{k=p+1}^{n_1} a(\lambda_\alpha, \xi_{i_k}) (\mathcal{B}^-_p)_{\alpha\beta} \quad (1 \leq \alpha \leq p, m < \beta \leq p), \\ ((\mathcal{B}^-)^{(m)}_p)_{\alpha\beta} &= \mathcal{M}^-_{\alpha\beta} \quad (1 \leq \alpha \leq p, 1 \leq \beta < m), \\ ((\mathcal{B}^-)^{(m)}_p)_{\alpha m} &= \prod_{k=p+1}^{n_1} a(\lambda_\alpha, \xi_{i_k}) \sum_{j \neq i_{m+1}, \dots, i_{n_1}} b^+(\mu_m, \xi_j) b^-(\lambda_\alpha, \xi_j) \prod_{\gamma=1}^{\alpha-1} a(\lambda_\gamma, \xi_j) \\ & \times \prod_{\epsilon=1}^{\beta-1} a(\mu_\epsilon, \xi_j) \prod_{k \neq j, i_{p+1}, \dots, i_{n_1}}^N a^{-1}(\xi_k, \xi_j) \quad (1 \leq \alpha \leq p). \end{aligned} \quad (6.40)$$

By the procedure leading to $((\mathcal{B}^-)^{(1)}_p)_{\alpha\beta}$, we prove $((\mathcal{B}^-)^{(m)}_p)_{\alpha m} = \mathcal{M}^-_{\alpha m}$. Therefore we have proved that the function (6.37) holds for all $m \leq p$.

26 *Shao-You Zhao et al*

When $m = p$, we have,

$$\begin{aligned} & G^{(p)}(\{\lambda_k\}_{(p,n_1)}, \mu_1, \dots, \mu_p, i_{p+1}, \dots, i_{n_1}) \\ &= (-1)^p (2 \cosh \eta)^{\frac{(n_1-p)(n_1-p-1)}{2}} \det \mathcal{M}^-(\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_p) \\ & \times \prod_{l=1}^p \prod_{k=p+1}^{n_1} a(\mu_l, \xi_{i_k}) \det \mathcal{B}_{n_1-p}^-(\lambda_{p+1}, \dots, \lambda_{n_1}; \xi_{i_{p+1}}, \dots, \xi_{i_{n_1}}), \end{aligned} \quad (6.41)$$

where the matrix elements of \mathcal{M}^- are given by (6.36).

For later use, we rewrite the element of the matrix $\mathcal{M}_{\alpha\beta}^\pm$ ($1 \leq \alpha, \beta \leq p$) in the form

$$\begin{aligned} \mathcal{M}_{\alpha\beta}^\pm &= \mathcal{F}_{\alpha\beta}^\pm + \sum_{\epsilon=1}^{\beta} \prod_{j=p+1}^{n_1} (a^{-1}(\mu_\epsilon, \xi_{i_j}) a(\lambda_\alpha, \xi_{i_j})) (\mathcal{G}^\pm)_{\alpha\beta}^\epsilon \\ &+ \sum_{\gamma=1}^{\alpha-1} \prod_{j=p+1}^{n_1} (a^{-1}(\lambda_\gamma, \xi_{i_j}) a(\lambda_\alpha, \xi_{i_j})) (\mathcal{H}^\pm)_{\alpha\beta}^\gamma, \end{aligned} \quad (6.42)$$

where

$$\begin{aligned} \mathcal{F}_{\alpha\beta}^\pm &= \frac{b^\pm(\lambda_\alpha, \mu_\beta)}{a(\lambda_\alpha, \mu_\beta)} \prod_{\gamma=1}^{\alpha-1} a^{-1}(\mu_\beta, \lambda_\gamma) \prod_{\epsilon=1}^{\beta-1} a^{-1}(\mu_\beta, \mu_\epsilon) \prod_{j=p+1}^{n_1} a(\mu_\beta, \lambda_j^{(1)}) \\ &+ \sum_{j=p+1}^{n_1} \left[b^\mp(\mu_\beta, \lambda_j^{(1)}) b^\pm(\lambda_\alpha, \lambda_j^{(1)}) \prod_{\gamma=1}^{\alpha-1} a(\lambda_\gamma, \lambda_j^{(1)}) \prod_{\epsilon=1}^{\beta-1} a(\mu_\epsilon, \lambda_j^{(1)}) \prod_{k \neq j} a^{-1}(\lambda_k^{(1)}, \lambda_j^{(1)}) \right] \\ &- \sum_{\gamma=1}^{\alpha-1} \frac{b^\pm(\lambda_\alpha, \lambda_\gamma)}{a(\lambda_\alpha, \lambda_\gamma)} \frac{b^\mp(\mu_\beta, \lambda_\gamma)}{a(\mu_\beta, \lambda_\gamma)} \prod_{\iota=1, \neq \gamma}^{\alpha-1} a^{-1}(\lambda_\gamma, \lambda_\iota) \prod_{\epsilon=1}^{\beta-1} a^{-1}(\lambda_\gamma, \mu_\epsilon) \prod_{j=p+1}^{n_1} a(\lambda_\gamma, \lambda_j^{(1)}) \\ &- \sum_{\epsilon=1}^{\beta-1} \frac{b^\pm(\lambda_\alpha, \mu_\epsilon)}{a(\lambda_\alpha, \mu_\epsilon)} \frac{b^\mp(\mu_\beta, \mu_\epsilon)}{a(\mu_\beta, \mu_\epsilon)} \prod_{\gamma=1}^{\alpha-1} a^{-1}(\mu_\epsilon, \lambda_\gamma) \prod_{\varepsilon=1, \neq \epsilon}^{\beta-1} a^{-1}(\mu_\epsilon, \mu_\varepsilon) \prod_{j=p+1}^{n_1} a(\mu_\epsilon, \lambda_j^{(1)}), \\ (\mathcal{G}^\pm)_{\alpha\beta}^\epsilon &= \begin{cases} \frac{b^\pm(\lambda_\alpha, \mu_\epsilon)}{a(\lambda_\alpha, \mu_\epsilon)} \prod_{\gamma=1}^{\alpha-1} a^{-1}(\mu_\epsilon, \lambda_\gamma) \prod_{k=1}^N a(\mu_\beta, \xi_k) & (\epsilon = \beta) \\ \frac{b^\pm(\lambda_\alpha, \mu_\epsilon)}{a(\lambda_\alpha, \mu_\epsilon)} \frac{b^\mp(\mu_\beta, \mu_\epsilon)}{a(\mu_\beta, \mu_\epsilon)} \prod_{\gamma=1}^{\alpha-1} a^{-1}(\mu_\epsilon, \lambda_\gamma) \prod_{\varepsilon=1, \neq \epsilon}^{\beta-1} a^{-1}(\mu_\epsilon, \mu_\varepsilon) \prod_{k=1}^N a(\mu_\epsilon, \xi_k) & (1 \leq \epsilon \leq \beta-1) \end{cases} \quad (6.43) \\ (\mathcal{H}^\pm)_{\alpha\beta}^\gamma &= \frac{b^\pm(\lambda_\alpha, \lambda_\gamma)}{a(\lambda_\alpha, \lambda_\gamma)} \frac{b^\mp(\mu_\beta, \lambda_\gamma)}{a(\mu_\beta, \lambda_\gamma)} \prod_{\iota=1, \neq \gamma}^{\alpha-1} a^{-1}(\lambda_\gamma, \lambda_\iota) \prod_{\epsilon=1}^{\beta-1} a^{-1}(\lambda_\gamma, \mu_\epsilon) \prod_{k=1}^N a(\lambda_\gamma, \xi_k) \\ & \quad (1 \leq \gamma \leq \alpha-1). \end{aligned} \quad (6.44)$$

After a tedious computation similar to that for the supersymmetric t - J model³⁷, we obtain the determinant of the matrix \mathcal{M}

$$\det \mathcal{M}^\pm(\{\lambda_\alpha\}, \{\mu_\beta\})$$

$$\begin{aligned}
&= \det \mathcal{F}^\pm(\{\lambda_\alpha\}, \{\mu_\beta\}) \\
&+ \sum_{k, l_k, \varrho_{m_k}^k} ' \prod_{e=1}^k \prod_{g=p+1}^{n_1} a^{-l_e}(\mu_e, \xi_{i_g}) \sum_{\substack{j_1, \dots, j_p=1 \\ j_1 \neq \dots \neq j_p}}^p (-1)^{\tau(j_1 j_2 \dots j_p)} \\
&\times \prod_{f'=1}^p \prod_{f=p+1}^{n_1} \prod_{t=1}^k \prod_{t'=1}^{l_t} \left[1 + \delta_{j_{f'}, \varrho_{t'}^{t'}} (a(\lambda_{f'}, \xi_{i_f}) - 1) \right] \\
&\times (A_2^\pm)_{1 j_1} (A_2^\pm)_{2 j_2} \dots (A_2^\pm)_{p j_p} \\
&+ \sum_{k, l_k, \rho_{m_k}^k} '' \prod_{e=1}^k \prod_{g=p+1}^{n_1} a^{-l_e}(\lambda_e, \xi_{i_g}) \prod_{t=1}^k \prod_{t'=1}^{l_t} \prod_{g'=p+1}^{n_1} a(\lambda_{\rho_{t'}^{t'}}, \xi_{i_{g'}}) \det A_3^\pm(\{\lambda_\alpha\}, \{\mu_\beta\}) \\
&+ \sum_{k, l_k, \varrho_{m_k}^k} ' \sum_{k', l_{k'}, \rho_{m_{k'}}^{k'}} '' \prod_{e=1}^k \prod_{g=p+1}^{n_1} a^{-l'_e}(\lambda_e, \xi_{i_g}) \prod_{t=1}^{k'} \prod_{t'=1}^{l'_t} \prod_{g'=p+1}^{n_1} (a(\lambda_{\rho_{t'}^{t'}}, \xi_{i_{g'}})) \\
&\times \sum_{\substack{j_1, \dots, j_p=1 \\ j_1 \neq \dots \neq j_p}}^p (-1)^{\tau(j_1 j_2 \dots j_p)} \\
&\times \prod_{t=1}^k \prod_{f=p+1}^{n_1} \prod_{s=1}^p \prod_{s \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_k}^k\}} \prod_{f'=1}^p \prod_{t'=1}^{l_t} \left[1 + \delta_{f', s} \delta_{j_{f'}, \varrho_{t'}^{t'}} (a^{-1}(\mu_t, \xi_{i_f}) - 1) \right] \\
&\times \prod_{t=1}^k \prod_{f=p+1}^{n_1} \prod_{s=1}^p \prod_{s \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_k}^k\}} \prod_{f'=1}^p \prod_{t'=1}^{l_t} \left[1 + \delta_{f', s} \delta_{j_{f'}, \varrho_{t'}^{t'}} (a(\lambda_s, \xi_{i_f}) - 1) \right] \\
&\times (A_4^\pm)_{1 j_1} (A_4^\pm)_{2 j_2} \dots (A_4^\pm)_{p j_p} \tag{6.45} \\
&\equiv \mathcal{T}_1^\pm + \mathcal{T}_2^\pm + \mathcal{T}_3^\pm + \mathcal{T}_4^\pm, \tag{6.46}
\end{aligned}$$

where $\tau(x_1, \dots, x_p) = \tau(\sigma)$, ($\sigma \in \mathcal{S}_p$ and $(x_1, \dots, x_p) = \sigma(1, \dots, p)$), $\tau(\sigma) = 0$ if σ is even and $\tau(\sigma) = 1$ if σ is odd,

$$\begin{aligned}
\sum_{k, l_k, \varrho_{m_k}^k} ' &= \sum_{k=1}^p \left\{ \sum_{l_k=1}^{p-k+1} \frac{1}{l_k!} \prod_{m_k=1}^{l_k} \sum_{\substack{\varrho_{m_k}^k = k \\ \varrho_{m_k}^k \neq \varrho_{m_k-1}^k, \dots, \varrho_1^k}}^p \right\} \left\{ \prod_{r=1}^{k-1} \sum_{l_{k-r}=0}^{p-k+r+1-\sum_{j=0}^{r-1} l_{k-j}} \frac{1}{l_{k-r}!} \right. \\
&\times \left. \prod_{m_{k-r}=1}^{l_{k-r}} \sum_{\substack{\varrho_{m_{k-r}}^{k-r} = k-r \\ \varrho_{m_{k-r}}^{k-r} \neq \varrho_{m_{k-r}-1}^{k-r}, \dots, \varrho_1^{k-r} \\ \varrho_{m_{k-r}}^{k-r} \neq \{\varrho_{k-r+1}^{k-r+1}\}, \{\varrho_{k-r+2}^{k-r+2}\}, \dots, \{\varrho_k^k\}}}^p \right\},
\end{aligned}$$

28 *Shao-You Zhao et al*

$$\sum_{k, l_k, \rho_{m_k}^k}'' = \sum_{k=1}^{p-1} \left\{ \sum_{l_k=1}^{p-k} \frac{1}{l_k!} \prod_{m_k=1}^{l_k} \sum_{\substack{\rho_{m_k}^k = k+1 \\ \rho_{m_k}^k \neq \rho_{m_k-1}^k, \dots, \rho_1^k}}^p \right\} \left\{ \prod_{r=1}^{k-1} \sum_{l_{k-r}=0}^{p-k+r-\sum_{j=0}^{r-1} l_{k-j}} \frac{1}{l_{k-r}!} \right. \\ \left. \times \prod_{m_{k-r}=1}^{l_{k-r}} \sum_{\substack{\rho_{m_{k-r}}^{k-r} = k-r+1 \\ \rho_{m_{k-r}}^{k-r} \neq \rho_{m_{k-r}-1}^{k-r}, \dots, \rho_1^{k-r} \\ \rho_{m_{k-r}}^{k-r} \neq \{\rho_{m_{k-r}-1}^{k-r+1}, \{\rho_{m_{k-r}-2}^{k-r+2}\}, \dots, \{\rho_{m_{k-r}-r}^{k-r+1}\}\}}}^p \right\},$$

and the elements $(A_i^\pm)_{\alpha\beta}$, $i = 2, 3, 4$, are given by

$$(A_2^\pm)_{\alpha\beta} = \begin{cases} \mathcal{F}_{\alpha\beta}^\pm & \alpha = 1, \dots, p, \beta = 1, \dots, p, \beta \neq \{\varrho_{m_1}^1\}, \dots, \{\varrho_{m_{n_1}}^{n_1}\} \\ (\mathcal{G}^\pm)_{\alpha\beta}^k & \alpha = 1, \dots, p, \beta = \{\varrho_{m_k}^k\} (k = 1, 2, \dots, p) \end{cases}, \quad (6.47)$$

$$(A_3^\pm)_{\alpha\beta} = \begin{cases} \mathcal{F}_{\alpha\beta}^\pm & \alpha = 1, \dots, p, \alpha \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_{n_1-1}}^{n_1-1}\}, \beta = 1, \dots, p \\ \frac{1}{\beta+1} (\mathcal{H}^\pm)_{\alpha\beta}^k & \alpha = \{\rho_{m_k}^k\} (k = 1, 2, \dots, p-1), \beta = 1, \dots, p \end{cases}, \quad (6.48)$$

and

$$(A_4^\pm)_{\alpha\beta} = \begin{cases} \mathcal{F}_{\alpha\beta}^\pm & \alpha = 1, \dots, p, \alpha \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_{n_1-1}}^{n_1-1}\}, \\ & \beta = 1, \dots, p, \beta \neq \{\varrho_{m_1}^1\}, \dots, \{\varrho_{m_{n_1}}^{n_1}\} \\ (\mathcal{G}^\pm)_{\alpha\beta}^k & \alpha = 1, \dots, p, \alpha \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_{n_1-1}}^{n_1-1}\}, \\ & \beta = \{\varrho_{m_k}^k\} (k = 1, 2, \dots, n) \\ \frac{1}{\beta+1} (\mathcal{H}^\pm)_{\alpha\beta}^k & \alpha = \{\rho_{m_k}^k\} (k = 1, \dots, n_1-1), \beta = 1, \dots, p \end{cases}, \quad (6.49)$$

respectively.

Thus by using (6.46), the function $G^{(p)}$ (6.41) becomes

$$\begin{aligned} & G^{(p)}(\{\lambda_k\}_{(p, n_1)}, \mu_1, \dots, \mu_p, i_{p+1}, \dots, i_{n_1}) \\ &= (-1)^p (2 \cosh \eta)^{\frac{(n_1-p)(n_1-p-1)}{2}} \sum_{j=1}^4 \mathcal{T}_j^- \\ & \quad \times \prod_{l=1}^p \prod_{k=p+1}^{n_1} a(\mu_l, \xi_{i_k}) \det \mathcal{B}_{n_1-p}^-(\lambda_{p+1}, \dots, \lambda_{n_1}; \xi_{i_{p+1}}, \dots, \xi_{i_{n_1}}) \end{aligned}$$

$$\equiv \sum_{j=1}^4 (G^-)_j^{(p)}(\{\lambda_k\}_{(p,n_1)}, \mu_1, \dots, \mu_p, i_{p+1}, \dots, i_{n_1}). \quad (6.50)$$

- $m \geq p+1$

Then we compute the intermediate functions $G^{(m)}$ for $m \geq p+1$. Similar to the $m \leq p$ case, inserting a complete set and noticing (6.50), we have

$$\begin{aligned} & G^{(m)}(\{\lambda_k\}_{(p,n_1)}, \mu_1, \dots, \mu_m, i_{m+1}, \dots, i_{n_1}) \\ &= \sum_{j \neq i_{m+1}, \dots, i_{n_1}}^N \langle 0 | \prod_{k=m+1}^{\overleftarrow{n_1}} E_{(i_k)}^{31} \tilde{B}_1(\mu_m) \prod_{k=m+1}^{\overrightarrow{m+q}} E_{(i_k)}^{13} E_{(j)}^{13} \prod_{m+q+1}^{\overrightarrow{n_1}} E_{(i_k)}^{13} | 0 \rangle \\ & \quad \times (G^-)^{(m-1)}(\{\lambda_k\}_{(p,n_1)}, \mu_1, \dots, \mu_{m-1}, i_{m+1}, \dots, i_{m+q}, j, i_{m+q+1}, \dots, i_{n_1}) \\ &= \sum_{j=1}^4 (G^-)_j^{(m)}(\{\lambda_k\}_{(p,n_1)}, \mu_1, \dots, \mu_m, i_{m+1}, \dots, i_{n_1}), \end{aligned} \quad (6.51)$$

where $(G^-)_j^{(m)}$'s correspond to $(G^-)_j^{(p)}$'s in (6.50), respectively.

We first compute $(G^-)_1^{(m)}$. With the help of the expression of \tilde{B}_1 (5.25), we have

$$\begin{aligned} & \langle 0 | \prod_{k=m+1}^{\overleftarrow{n_1}} E_{(i_k)}^{31} \tilde{B}_1(\mu_m) \prod_{k=m+1}^{\overrightarrow{m+q}} E_{(i_k)}^{13} E_{(j)}^{13} \prod_{m+q+1}^{\overrightarrow{n_1}} E_{(i_k)}^{13} | 0 \rangle \\ &= -(-1)^q (2 \cosh \eta)^{-(n_1-m)} b^+(\mu_m, \xi_j) \prod_{l=m+1}^{n_1} a(\mu_m, \xi_{i_l}) \prod_{k \neq j}^N a^{-1}(\xi_k, \xi_j). \end{aligned} \quad (6.52)$$

When $m = p+1$, by using the expressions (6.50) and (6.52), the intermediate function $(G^-)_1^{(p+1)}$ is given by

$$\begin{aligned} & (G^-)_1^{(p+1)}(\{\lambda_k\}_{(p,n_1)}, \mu_1, \dots, \mu_{p+1}, i_{p+2}, \dots, i_{n_1}) \\ &= \sum_{j \neq i_{p+2}, \dots, i_{n_1}}^N \langle 0 | \prod_{k=p+2}^{\overleftarrow{n_1}} E_{(i_k)}^{31} \tilde{B}_1(\mu_{p+1}) \prod_{k=p+2}^{\overrightarrow{p+q+1}} E_{(i_k)}^{13} E_{(j)}^{13} \prod_{p+q+2}^{\overrightarrow{n_1}} E_{(i_k)}^{13} | 0 \rangle \\ & \quad \times (G^-)_1^{(p)}(\{\lambda_k\}_{(p,n_1)}, \mu_1, \dots, \mu_p, i_{p+2}, \dots, i_{p+q+1}, j, i_{p+q+2}, \dots, i_{n_1}) \\ &= (-1)^m (2 \cosh \eta)^{\frac{(n_1-p)(n_1-p-1)-2(n_1-p-1)}{2}} \det \mathcal{F}(\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_p) \\ & \quad \times \det(\mathcal{B}^-)_{n_1-p}^{(p+1)}(\lambda_{p+1}, \dots, \lambda_{n_1}; \mu_{p+1}; \xi_{i_{p+2}}, \dots, \xi_{i_{n_1}}), \end{aligned} \quad (6.53)$$

where the matrix elements $((\mathcal{B}^-)_{n_1-p}^{(m)})_{\alpha\beta}$ ($p+1 \leq \alpha, \beta \leq n_1$)

$$\left((\mathcal{B}^-)_{n_1-p}^{(p+1)} \right)_{\alpha\beta} = \prod_{\epsilon=1}^{\beta-1} a(\mu_\epsilon, \xi_{i_\beta}) (\mathcal{B}^-)_{n_1-p} \alpha\beta, \quad \text{for } p+1 < \beta \leq n_1,$$

30 *Shao-You Zhao et al*

$$\begin{aligned}
\left((\mathcal{B}^-)^{(p+1)}\right)_{\alpha p+1} &= \sum_{j \neq i_{p+2}, \dots, i_{n_1}}^N b^+(\mu_{p+1}, \xi_j) b^-(\lambda_\alpha, \xi_j) \prod_{\gamma=p+1}^{\alpha-1} a(\lambda_\gamma, \xi_j) \\
&\quad \times \prod_{\epsilon=1}^p a(\mu_\epsilon, \xi_j) \prod_{k \neq j}^N a^{-1}(\xi_k, \xi_j). \tag{6.54}
\end{aligned}$$

Then by using similar procedure as the $m \leq p$ case, one prove that $\left((\mathcal{B}^\pm)^{(p+1)}\right)_{\alpha p+1}$ is equal to the function $(\mathcal{N}_1)_{\alpha\beta}$ ($p+1 \leq \alpha, \beta \leq n_1$)

$$\begin{aligned}
(\mathcal{N}_1^\pm)_{\alpha\beta} &= \frac{b^\pm(\lambda_\alpha, \mu_\beta)}{a(\lambda_\alpha, \mu_\beta)} \prod_{\gamma=p+1}^{\alpha-1} a^{-1}(\mu_\beta, \lambda_\gamma) \prod_{\epsilon=1}^{\beta-1} a^{-1}(\mu_\beta, \mu_\epsilon) \\
&\quad \times \left[\prod_{j=p+1}^{n_1} a(\mu_\beta, \lambda_j^{(1)}) - \prod_{k=1}^N a(\mu_\beta, \xi_k) \right] \\
&\quad + \sum_{j=p+1}^{n_1} \left[b^\mp(\mu_\beta, \lambda_j^{(1)}) b^\pm(\lambda_\alpha, \lambda_j^{(1)}) \prod_{\gamma=p+1}^{\alpha-1} a(\lambda_\gamma, \lambda_j^{(1)}) \right. \\
&\quad \times \left. \prod_{\epsilon=1}^{\beta-1} a(\mu_\epsilon, \lambda_j^{(1)}) \prod_{k \neq j=p+1}^{n_1} a^{-1}(\lambda_k^{(1)}, \lambda_j^{(1)}) \right] \\
&\quad - \sum_{\epsilon=1}^{\beta-1} \frac{b^\pm(\lambda_\alpha, \mu_\epsilon)}{a(\lambda_\alpha, \mu_\epsilon)} \prod_{\gamma=p+1}^{\alpha-1} a^{-1}(\mu_\epsilon, \lambda_\gamma) \prod_{\varepsilon=1, \neq \epsilon}^{\beta-1} a^{-1}(\mu_\epsilon, \mu_\varepsilon) \\
&\quad \times \left[\prod_{j=p+1}^{n_1} a(\mu_\epsilon, \lambda_j^{(1)}) - \prod_{k=1}^N a(\mu_\epsilon, \xi_k) \right]. \tag{6.55}
\end{aligned}$$

Moreover, with a similar procedure, one may prove that for any $p+1 \leq m \leq n_1$, the function $(G^-)_1^{(m)}$ can be written as

$$\begin{aligned}
&(G^-)_1^{(m)}(\{\lambda_k\}_{(p, n_1)}, \mu_1, \dots, \mu_m, i_{m+1}, \dots, i_{n_1}) \\
&= (-1)^m 2^{\frac{(n_1-p)(n_1-p-1)-(m-p)(2n_1-m-p-1)}{2}} \det \mathcal{F}^-(\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_p) \\
&\quad \times \det(\mathcal{B}^-)_{n_1-p}^{(m)}(\lambda_{p+1}, \dots, \lambda_{n_1}; \xi_{i_{p+1}}, \dots, \xi_{i_{n_1}}), \tag{6.56}
\end{aligned}$$

where the matrix elements $((\mathcal{B}^-)_{n_1-p}^{(m)})_{\alpha\beta}$ ($p+1 \leq \alpha, \beta \leq n_1$)

$$\begin{aligned}
((\mathcal{B}^-)_{n_1-p}^{(m)})_{\alpha\beta} &= \prod_{\epsilon=1}^{\beta-1} a(\mu_\epsilon, \xi_{i_\beta}) (\mathcal{B}_p^-)_{\alpha\beta}, \quad \text{for } m < \beta \leq n_1, \\
((\mathcal{B}^-)_{n_1-p}^{(m)})_{\alpha\beta} &= (\mathcal{N}_1^-)_{\alpha\beta}, \quad \text{for } p+1 \leq \beta \leq m. \tag{6.57}
\end{aligned}$$

Therefore when $m = n_1$, we obtain

$$\begin{aligned}
(G^-)_1^{(n_1)}(\{\lambda_j\}_{(p, n_1)}, \{\mu_k\}_{(p, n_1)}) &= (-1)^{n_1} \det \mathcal{F}^-(\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_p) \\
&\quad \times \det \mathcal{N}_1^-(\lambda_{p+1}, \dots, \lambda_{n_1}; \mu_{p+1}, \dots, \mu_{n_1}). \tag{6.58}
\end{aligned}$$

Similarly, the function $(G^-)^{(n_1)}_2$ is given by

$$\begin{aligned} (G^-)^{(n_1)}_2(\{\lambda_j\}_{(p,n_1)}, \{\mu_k\}_{(p,n_1)}) &= (-1)^{n_1} \sum_{k,l_k,\varrho_{m_k}^k} ' \sum_{\substack{j_1,\dots,j_p=1 \\ j_1 \neq \dots \neq j_p}}^p (-1)^{\tau(j_1 j_2 \dots j_p)} \\ &\times (A_2^-)_{1j_1} (A_2^-)_{2j_2} \dots (A_2^-)_{pj_p} \det \mathcal{N}_2^-(\lambda_{p+1}, \dots, \lambda_{n_1}; \mu_{p+1}, \dots, \mu_{n_1}; \{l_k\}) \end{aligned} \quad (6.59)$$

with

$$\begin{aligned} (\mathcal{N}_2^\pm)_{\alpha\beta} &= \frac{b^\pm(\lambda_\alpha, \mu_\beta)}{a(\lambda_\alpha, \mu_\beta)} \prod_{e=1}^k a^{l_e-1}(\mu_\beta, \mu_e) \prod_{\epsilon=k+1}^{\beta-1} a^{-1}(\mu_\beta, \mu_\epsilon) \prod_{\gamma=p+1}^{\alpha-1} a^{-1}(\mu_\beta, \lambda_\gamma) \\ &\times \prod_{f'=1}^p \prod_{t=1}^k \prod_{t'=1}^{l_t} \left[1 + \delta_{j_{f'}, \varrho_{t'}^t} (a^{-1}(\mu_\beta, \lambda_{f'}) - 1) \right] \left[\prod_{j=p+1}^{n_1} a(\mu_\beta, \lambda_j^{(1)}) - \prod_{l=1}^N a(\mu_\beta, \xi_l) \right] \\ &+ \sum_{\theta=p+1}^{n_1} b^\mp(\mu_\beta, \lambda_\theta^{(1)}) b^\pm(\lambda_\alpha, \lambda_\theta^{(1)}) \prod_{e=1}^k a^{l_e-1}(\mu_e, \lambda_\theta^{(1)}) \prod_{\epsilon=k+1}^{\beta-1} a(\mu_e, \lambda_\theta^{(1)}) \\ &\times \prod_{\gamma=p+1}^{\alpha-1} a(\lambda_\gamma, \lambda_\theta^{(1)}) \prod_{\vartheta=p+1, \neq \theta}^{n_1} a^{-1}(\lambda_\vartheta, \lambda_\theta^{(1)}) \\ &\times \prod_{f'=1}^p \prod_{t=1}^k \prod_{t'=1}^{l_t} \left[1 + \delta_{j_{f'}, \varrho_{t'}^t} (a(\lambda_{f'}, \lambda_\theta^{(1)}) - 1) \right] \\ &- \sum_{\epsilon=k+1}^{\beta-1} \frac{b^\mp(\mu_\beta, \mu_\epsilon)}{a(\mu_\beta, \mu_\epsilon)} \frac{b^\pm(\lambda_\alpha, \mu_\epsilon)}{a(\lambda_\alpha, \mu_\epsilon)} \prod_{e=1}^k a^{l_e-1}(\mu_\epsilon, \mu_e) \prod_{\varepsilon=k+1, \neq \epsilon}^{\beta-1} a^{-1}(\mu_\epsilon, \mu_\varepsilon) \\ &\times \prod_{\gamma=p+1}^{\alpha-1} a^{-1}(\mu_\epsilon, \lambda_\gamma) \left[\prod_{j=p+1}^{n_1} a(\mu_\epsilon, \lambda_j^{(1)}) - \prod_{l=1}^N a(\mu_\epsilon, \xi_l) \right] \\ &\times \prod_{f'=1}^p \prod_{t=1}^k \prod_{t'=1}^{l_t} \left[1 + \delta_{j_{f'}, \varrho_{t'}^t} (a^{-1}(\mu_\epsilon, \lambda_{f'}) - 1) \right] \\ &+ \sum_{e=1}^k g_2^\pm(\mu_\beta, l_e), \end{aligned} \quad (6.60)$$

where the function $g_2^\pm(\mu_\beta, l_e) = 0$ when $l_e = 1$; when $l_e = 0$,

$$\begin{aligned} g_2^\pm(\mu_\beta, l_e) &= - \frac{b^\mp(\mu_\beta, \mu_e)}{a(\mu_\beta, \mu_e)} \frac{b^\pm(\lambda_\alpha, \mu_e)}{a(\lambda_\alpha, \mu_e)} \prod_{e'=1, \neq e}^k a^{l_{e'}-1}(\mu_e \mu_{e'}) \prod_{\epsilon=k+1}^{\beta-1} a^{-1}(\mu_e - \mu_\epsilon) \\ &\times \prod_{\gamma=p+1}^{\alpha-1} a^{-1}(\mu_e, \lambda_\gamma) \left[\prod_{j=p+1}^{n_1} a(\mu_e, \lambda_j^{(1)}) - \prod_{l=1}^N a(\mu_e, \xi_l) \right] \end{aligned}$$

32 *Shao-You Zhao et al*

$$\times \prod_{f'=1}^p \prod_{t=1}^k \prod_{t'=1}^{l_t} \left[1 + \delta_{j_{f'}, e_{t'}^t} (a(\mu_e, \lambda_{f'}) - 1) \right]; \quad (6.61)$$

when $l_e \geq 2$,

$$\begin{aligned} g_2(\mu_\beta, l_e) = & - \sum_{k=0}^{l_e-2} \frac{1}{k!} \frac{e^{\mp \mu_\beta}}{\sinh(\mu_\beta - \mu_e + \eta)^{l_e-k-1}} \frac{d^k}{d\chi_2^k} \left\{ \sinh^{l_e-1}(\mu_\beta - \mu_e) \right. \\ & \times \frac{e^{\pm \mu_\beta} b^\pm(\lambda_\alpha, \mu_\beta) \eta}{a(\lambda_\alpha, \mu_\beta)} \prod_{e'=1, \neq e}^k a^{l_{e'}-1}(\mu_\beta, \mu_{e'}) \prod_{\epsilon=k+1}^{\beta-1} a^{-1}(\mu_\beta, \mu_\epsilon) \prod_{\gamma=p+1}^{\alpha-1} a^{-1}(\mu_\beta, \lambda_\gamma) \\ & \times \prod_{f'=1}^p \prod_{t=1}^k \prod_{t'=1}^{l_t} \left[1 + \delta_{j_{f'}, e_{t'}^t} (a^{-1}(\mu_\beta, \lambda_{f'}) - 1) \right] \\ & \times \left[\prod_{j=p+1}^{n_1} a(\mu_\beta, \lambda_j^{(1)}) - \prod_{l=1}^N a(\mu_\beta, \xi_l) \right], \left. \right\}_{\chi_2=0} \end{aligned} \quad (6.62)$$

and $\chi_2 \equiv \chi_2(\mu_\beta) = \sinh(\mu_\beta - \mu_e + \eta)$.The function $(G^-)_3^{(n_1)}$ is given by

$$\begin{aligned} & (G^-)_3^{(n_1)}(\{\lambda_j\}_{(p, n_1)}, \{\mu_k\}_{(p, n_1)}) \\ & = (-1)^{n_1} \sum_{k, l_k, \rho_{m_k}^k}'' \det A_3^-(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(p)}; \mu_{\sigma'(1)}, \dots, \mu_{\sigma'(p)}) \\ & \quad \times \det \mathcal{N}_3^-(\lambda_{p+1}, \dots, \lambda_{n_1}; \mu_{p+1}, \dots, \mu_{n_1}; \{l_k\}) \end{aligned} \quad (6.63)$$

with

$$\begin{aligned} (\mathcal{N}_3^\pm)_{\alpha\beta} = & \frac{b^\pm(\lambda_\alpha, \mu_\beta)}{a(\lambda_\alpha, \mu_\beta)} \prod_{e=1}^k a^{l_e}(\mu_\beta, \lambda_e) \prod_{t=1}^k \prod_{t'=1}^{l_t} a^{-1}(\mu_\beta, \lambda_{\rho_{t'}^t}) \prod_{\gamma=p+1}^{\alpha-1} a^{-1}(\mu_\beta, \lambda_\gamma) \\ & \times \prod_{\epsilon=1}^{\beta-1} a^{-1}(\mu_\beta, \mu_\epsilon) \left[\prod_{j=p+1}^{n_1} a(\mu_\beta, \lambda_j^{(1)}) - \prod_{l=1}^N a(\mu_\beta, \xi_l) \right] \\ & + \sum_{\theta=p+1}^{n_1} b^\mp(\mu_\beta, \lambda_\theta^{(1)}) b^\pm(\lambda_\alpha, \lambda_\theta^{(1)}) \prod_{\epsilon=1}^{\beta-1} a(\mu_\epsilon, \lambda_\theta^{(1)}) \prod_{\gamma=p+1}^{\alpha-1} a(\lambda_\gamma, \lambda_\theta^{(1)}) \\ & \times \prod_{e=1}^k a^{-l_e}(\mu_e, \lambda_\theta^{(1)}) \prod_{t=1}^k \prod_{t'=1}^{l_t} a(\lambda_{\rho_{t'}^t}, \lambda_\theta^{(1)}) \prod_{\vartheta=p+1, \neq \theta}^{n_1} a^{-1}(\lambda_\vartheta, \lambda_\theta^{(1)}) \\ & - \sum_{\epsilon=k+1}^{\beta-1} b^\mp(\mu_\beta, \mu_\epsilon) b^\pm(\lambda_\alpha, \mu_\epsilon) \prod_{e=1}^k a^{l_e}(\mu_\epsilon, \lambda_e) \prod_{t=1}^k \prod_{t'=1}^{l_t} a^{-1}(\lambda_{\mu_\epsilon, \rho_{t'}^t}) \\ & \times \prod_{\varepsilon=1, \neq \epsilon}^{\beta-1} a^{-1}(\mu_\epsilon, \mu_\varepsilon) \prod_{\gamma=p+1}^{\alpha-1} a^{-1}(\mu_\epsilon, \lambda_\gamma) \left[\prod_{j=p+1}^{n_1} a(\mu_\epsilon, \lambda_j^{(1)}) - \prod_{l=1}^N a(\mu_\epsilon, \xi_l) \right] \end{aligned}$$

$$+ \sum_{e=1}^k g_3^{\pm}(\mu_{\beta}, l_e), \quad (6.64)$$

where the function $g_3^{\pm}(\mu_{\beta}, l_e)$ is given as follows. i.) when $\prod_{t=1}^k \prod_{t'=1}^{l_t} \delta_{e \rho_{t'}^t} = 0$,

$$\begin{aligned} g_3(\mu_{\beta}, l_e) = & - \sum_{k=0}^{l_e-1} \frac{1}{k!} \frac{e^{\mp \mu_{\beta}}}{\sinh(\mu_{\beta} - \lambda_e + \eta)^{l_e-k}} \frac{d^k}{d\chi_3^k} \left\{ \sinh^{l_e}(\mu_{\beta} - \lambda_e) \frac{e^{\pm \mu_{\beta}} b^{\pm}(\lambda_{\alpha}, \mu_{\beta})}{a(\lambda_{\alpha}, \mu_{\beta})} \right. \\ & \times \prod_{e'=1, \neq e}^k a^{l_e}(\mu_{\beta}, \lambda_{e'}) \prod_{t=1}^k \prod_{t'=1}^{l_t} a^{-1}(\mu_{\beta}, \lambda_{\rho_{t'}^t}) \prod_{\gamma=p+1}^{\alpha-1} a^{-1}(\mu_{\beta}, \lambda_{\gamma}) \\ & \left. \times \prod_{\epsilon=1}^{\beta-1} a^{-1}(\mu_{\beta}, \mu_{\epsilon}) \left[\prod_{j=p+1}^{n_1} a(\mu_{\beta}, \lambda_j^{(1)}) - \prod_{l=1}^N a(\mu_{\beta}, \xi_l) \right] \right\}_{\chi_3=0}, \end{aligned} \quad (6.65)$$

and $\chi_3 \equiv \chi_3(\mu_{\beta}) = \sinh(\mu_{\beta} - \lambda_e + \eta)$, ii.) when $\prod_{t=1}^k \prod_{t'=1}^{l_t} \delta_{e \rho_{t'}^t} = 1$ and $l_e = 1$, $g_3^{\pm}(\mu_{\beta}, l_e) = 0$ and iii.) when there is an index \hat{t} ($\hat{t} \in \{1, \dots, k\}$) and \hat{t}' ($\hat{t}' \in \{1, \dots, l_{\hat{t}}\}$) such that $\rho_{\hat{t}'}^{\hat{t}} = e$, and $l_e \geq 2$,

$$\begin{aligned} g_3(\mu_{\beta}, l_e) = & - \sum_{k=0}^{l_e-2} \frac{1}{k!} \frac{e^{\mp \mu_{\beta}}}{\sinh(\mu_{\beta} - \lambda_e + \eta)^{l_e-k-1}} \frac{d^k}{d\chi_3^k} \left\{ \sinh^{l_e-1}(\mu_{\beta} - \lambda_e) \right. \\ & \times \frac{e^{\pm \mu_{\beta}} b^{\pm}(\lambda_{\alpha}, \mu_{\beta})}{a(\lambda_{\alpha}, \mu_{\beta})} \prod_{e'=1, \neq e}^k a^{l_e}(\mu_{\beta}, \lambda_{e'}) \prod_{t=1, \neq \hat{t}}^k \prod_{t'=1, \neq \hat{t}'}^{l_t} a^{-1}(\mu_{\beta}, \lambda_{\rho_{t'}^t}) \\ & \left. \times \prod_{\gamma=p+1}^{\alpha-1} a^{-1}(\mu_{\beta}, \lambda_{\gamma}) \prod_{\epsilon=1}^{\beta-1} a^{-1}(\mu_{\beta}, \mu_{\epsilon}) \left[\prod_{j=p+1}^{n_1} a(\mu_{\beta}, \lambda_j^{(1)}) - \prod_{l=1}^N a(\mu_{\beta}, \xi_l) \right] \right\}_{\chi_3=0}. \end{aligned} \quad (6.66)$$

The function $(G^-)^{(n_1)}_4$ is given by

$$\begin{aligned} & (G^-)^{(n_1)}_4(\{\lambda_j\}_{(p, n_1)}, \{\mu_k\}_{(p, n_1)}) \\ & = (-1)^{n_1} \sum_{k, l_k, \rho_{m_k}^k} ' \sum_{k', l'_{k'}, \rho_{m_{k'}}^{k'}} '' \sum_{\substack{j_1, \dots, j_p=1 \\ j_1 \neq \dots \neq j_p}}^p (-1)^{\tau(j_1 j_2 \dots j_p)} \\ & \times (A_4^-)_{1 j_1} (A_4^-)_{2 j_2} \dots (A_4^-)_{p j_p} \\ & \times \det \mathcal{N}_4^-(\lambda_{p+1}, \dots, \lambda_{n_1}; \mu_{p+1}, \dots, \mu_{n_1}; \{l_k\}; \{l'_{k'}\}) \end{aligned} \quad (6.67)$$

with

$$\begin{aligned} (\mathcal{N}_4^{\pm})_{\alpha\beta} = & \frac{b^{\pm}(\lambda_{\alpha}, \mu_{\beta})}{a(\lambda_{\alpha}, \mu_{\beta})} \prod_{e=1}^k a^{l'_e}(\mu_{\beta}, \lambda_e) \prod_{t=1}^{k'} \prod_{t'=1}^{l'_t} a^{-1}(\mu_{\beta}, \lambda_{\rho_{t'}^t}) \prod_{\gamma=p+1}^{\alpha-1} a^{-1}(\mu_{\beta}, \lambda_{\gamma}) \\ & \times \prod_{\epsilon=1}^{\beta-1} a^{-1}(\mu_{\beta}, \mu_{\epsilon}) \left[\prod_{j=p+1}^{n_1} a(\mu_{\beta}, \lambda_j^{(1)}) - \prod_{l=1}^N a(\mu_{\beta}, \xi_l) \right] \end{aligned}$$

34 *Shao-You Zhao et al*

$$\begin{aligned}
& \times \prod_{t=1}^k \prod_{\substack{s=1 \\ s \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_k}^k\}}}^p \prod_{f'=1}^p \prod_{t'=1}^{l_t} \left[1 + \delta_{f' s} \delta_{j_{f'} e_{t'}^t} (a(\mu_\beta, \mu_t) - 1) \right] \\
& \times \prod_{t=1}^k \prod_{\substack{s=1 \\ s \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_k}^k\}}}^p \prod_{f'=1}^p \prod_{t'=1}^{l_t} \left[1 + \delta_{f' s} \delta_{j_{f'} e_{t'}^t} (a^{-1}(\mu_\beta, \lambda_s) - 1) \right] \\
& + \sum_{\theta=p+1}^{n_1} b^\mp(\mu_\beta, \lambda_\theta^{(1)}) b^\pm(\lambda_\alpha, \lambda_\theta^{(1)}) \prod_{\epsilon=1}^{\beta-1} a(\mu_\epsilon, \lambda_\theta^{(1)}) \prod_{\gamma=p+1}^{\alpha-1} a(\lambda_\gamma, \lambda_\theta^{(1)}) \\
& \times \prod_{e=1}^k a^{-l'_e}(\mu_e, \lambda_\theta^{(1)}) \prod_{t=1}^{k'} \prod_{t'=1}^{l'_t} a(\lambda_{\rho_{t'}^t}, \lambda_\theta^{(1)}) \prod_{\vartheta=p+1, \neq \theta}^{n_1} a^{-1}(\lambda_\vartheta, \lambda_\theta^{(1)}) \\
& \times \prod_{t=1}^k \prod_{\substack{s=1 \\ s \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_k}^k\}}}^p \prod_{f'=1}^p \prod_{t'=1}^{l_t} \left[1 + \delta_{f' s} \delta_{j_{f'} e_{t'}^t} (a^{-1}(\mu_t, \lambda_\theta^{(1)}) - 1) \right] \\
& \times \prod_{t=1}^k \prod_{\substack{s=1 \\ s \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_k}^k\}}}^p \prod_{f'=1}^p \prod_{t'=1}^{l_t} \left[1 + \delta_{f' s} \delta_{j_{f'} e_{t'}^t} (a(\lambda_s, \lambda_\theta^{(1)}) - 1) \right] \\
& - \sum_{\epsilon=k+1}^{\beta-1} \frac{b^\mp(\mu_\beta, \mu_\epsilon)}{a(\mu_\beta, \mu_\epsilon)} \frac{b^p m(\lambda_\alpha, \mu_\epsilon)}{a(\lambda_\alpha, \mu_\epsilon)} \prod_{e=1}^k a^{l'_e}(\mu_\epsilon, \lambda_e) \prod_{t=1}^{k'} \prod_{t'=1}^{l'_t} a^{-1}(\mu_\epsilon, \lambda_{\rho_{t'}^t}) \\
& \times \prod_{\varepsilon=1, \neq \epsilon}^{\beta-1} a^{-1}(\mu_\epsilon, \mu_\varepsilon) \prod_{\gamma=p+1}^{\alpha-1} a^{-1}(\mu_\epsilon, \lambda_\gamma) \left[\prod_{j=p+1}^{n_1} a(\mu_\epsilon, \lambda_j^{(1)}) - \prod_{l=1}^N a(\mu_\epsilon, \xi_l) \right] \\
& \times \prod_{t=1}^k \prod_{\substack{s=1 \\ s \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_k}^k\}}}^p \prod_{f'=1}^p \prod_{t'=1}^{l_t} \left[1 + \delta_{f' s} \delta_{j_{f'} e_{t'}^t} (a(\mu_\epsilon, \mu_t) - 1) \right] \\
& \times \prod_{t=1}^k \prod_{\substack{s=1 \\ s \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_k}^k\}}}^p \prod_{f'=1}^p \prod_{t'=1}^{l_t} \left[1 + \delta_{f' s} \delta_{j_{f'} e_{t'}^t} (a^{-1}(\mu_\epsilon, \lambda_s) - 1) \right] \\
& + \sum_{e=1}^{k'} g_4^\pm(\mu_\beta, l'_e) + \sum_{t=1}^k (g')_4^\pm(\mu_\beta, l_t), \tag{6.68}
\end{aligned}$$

where $g_4^\pm(\mu_\beta, l'_e)$ is given as follows. i.) when $\prod_{t=1}^{k'} \prod_{t'=1}^{l'_t} \delta_e \rho_{t'}^t = 0$,

$$g_4^\pm(\mu_\beta, l'_e) = - \sum_{k=0}^{l'_e-1} \frac{1}{k!} \frac{e^{\mp \mu_\beta}}{\sinh(\mu_\beta - \lambda_e + \eta)^{l'_e-k}} \frac{d^k}{d\chi_4^k} \left\{ \sinh^{l'_e}(\mu_\beta - \lambda_e) \right\}$$

$$\begin{aligned}
& \times \frac{e^{\pm\mu_\beta} b^\pm(\lambda_\alpha, \mu_\beta)}{a(\lambda_\alpha, \mu_\beta)} \prod_{e'=1, \neq e}^{k'} a^{l'_{e'}}(\mu_\beta, \lambda_e) \prod_{t=1}^{k'} \prod_{t'=1}^{l'_t} a^{-1}(\mu_\beta, \lambda_{\rho_{t'}^t}) \prod_{\gamma=p+1}^{\alpha-1} a^{-1}(\mu_\beta, \lambda_\gamma) \\
& \times \prod_{\epsilon=1}^{\beta-1} a^{-1}(\mu_\beta, \mu_\epsilon) \left[\prod_{j=p+1}^{n_1} a(\mu_\beta, \lambda_j^{(1)}) - \prod_{l=1}^N a(\mu_\beta, \xi_l) \right] \\
& \times \prod_{t=1}^k \prod_{\substack{s=1 \\ s \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_k}^k\}}}^p \prod_{f'=1}^p \prod_{t'=1}^{l_t} \left[1 + \delta_{f' s} \delta_{j_{f'} \rho_{t'}^t} (a(\mu_\beta, \mu_t) - 1) \right] \\
& \times \prod_{t=1}^k \prod_{\substack{s=1 \\ s \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_k}^k\}}}^p \prod_{f'=1}^p \prod_{t'=1}^{l_t} \left[1 + \delta_{f' s} \delta_{j_{f'} \rho_{t'}^t} (a^{-1}(\mu_\beta, \lambda_s) - 1) \right] \Bigg\}_{\chi_4=0},
\end{aligned} \tag{6.69}$$

and $\chi_4 \equiv \chi_4(\mu_\beta) = \sinh(\mu_\beta - \lambda_e + \eta)$, ii.) when $\prod_{t=1}^k \prod_{t'=1}^{l'_t} \delta_{e \rho_{t'}^t} = 1$ and $l'_e = 1$, $g_4^\pm(\mu_\beta, l_e) = 0$ and iii.) when there are indices \hat{t} ($\hat{t} \in \{1, \dots, k\}$) and \hat{t}' ($\hat{t}' \in \{1, \dots, l_{\hat{t}}\}$) such that $\rho_{\hat{t}'}^{\hat{t}} = e$, and $l_e \geq 2$,

$$\begin{aligned}
g_4^\pm(\mu_\beta, l_e) &= - \sum_{k=0}^{l_e-2} \frac{1}{k!} \frac{e^{\mp\mu_\beta}}{\sinh(\mu_\beta - \lambda_e + \eta)^{l_e-k-1}} \frac{d^k}{d\chi_4^k} \left\{ \sinh^{l_e-1}(\mu_\beta - \lambda_e) \right. \\
& \times \frac{e^{\pm\mu_\beta} b^\pm(\lambda_\alpha, \mu_\beta)}{a(\lambda_\alpha, \mu_\beta)} \prod_{e'=1, \neq e}^{k'} a^{l'_{e'}}(\mu_\beta, \lambda_e) \prod_{t=1, \neq \hat{t}}^{k'} \prod_{t'=1, \neq \hat{t}'}^{l'_t} a^{-1}(\mu_\beta, \lambda_{\rho_{t'}^t}) \\
& \times \prod_{\gamma=p+1}^{\alpha-1} a^{-1}(\mu_\beta, \lambda_\gamma) \prod_{\epsilon=1}^{\beta-1} a^{-1}(\mu_\beta, \mu_\epsilon) \left[\prod_{j=p+1}^{n_1} a(\mu_\beta, \lambda_j^{(1)}) - \prod_{l=1}^N a(\mu_\beta, \xi_l) \right] \\
& \times \prod_{t=1}^k \prod_{\substack{s=1 \\ s \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_k}^k\}}}^p \prod_{f'=1}^p \prod_{t'=1}^{l_t} \left[1 + \delta_{f' s} \delta_{j_{f'} \rho_{t'}^t} (a(\mu_\beta, \mu_t) - 1) \right] \\
& \times \prod_{t=1}^k \prod_{\substack{s=1 \\ s \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_k}^k\}}}^p \prod_{f'=1}^p \prod_{t'=1}^{l_t} \left[1 + \delta_{f' s} \delta_{j_{f'} \rho_{t'}^t} (a^{-1}(\mu_\beta, \lambda_s) - 1) \right] \Bigg\}_{\chi_4=0};
\end{aligned} \tag{6.70}$$

for the function $(g')_4^\pm(\mu_\beta, l_t)$, one has : i.) $(g')_4^\pm(\mu_\beta, l_t) = 0$ when

$$n_t \equiv \sum_{\substack{s=1 \\ s \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_k}^k\}}}^p \sum_{f'=1}^p \sum_{t'=1}^{l_t} \delta_{f' s} \delta_{j_{f'} \rho_{t'}^t} = 1,$$

36 *Shao-You Zhao et al*ii.) when $n_t = 0$,

$$\begin{aligned}
(g')_4^\pm(\mu_\beta, l_t) &= -\frac{b^\mp(\mu_\beta, \mu_t)}{a(\mu_\beta, \mu_t)} \frac{b^\pm(\lambda_\alpha - \mu_t)}{a(\lambda_\alpha, \mu_t)} \prod_{e=1}^k a^{l'_e}(\mu_t, \lambda_e) \prod_{t=1}^{k'} \prod_{t'=1}^{l'_t} a^{-1}(\mu_t, \lambda_{\rho_{t'}^t}) \\
&\times \prod_{\epsilon=1, \neq t}^{\beta-1} a^{-1}(\mu_t, \mu_\epsilon) \prod_{\gamma=p+1}^{\alpha-1} a^{-1}(\mu_t, \lambda_\gamma) \left[\prod_{j=p+1}^{n_1} a(\mu_t, \lambda_j^{(1)}) - \prod_{l=1}^N a(\mu_t, \xi_l) \right] \\
&\times \prod_{\tau=1, \neq t}^k \prod_{\substack{s=1 \\ s \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_k}^k\}}}^p \prod_{f'=1}^p \prod_{\tau'=1}^{l_\tau} \left[1 + \delta_{f' s} \delta_{j_{f'} \rho_{\tau'}^\tau} (a(\mu_t, \mu_\tau) - 1) \right] \\
&\times \prod_{\tau=1}^k \prod_{\substack{s=1 \\ s \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_k}^k\}}}^p \prod_{f'=1}^p \prod_{\tau'=1}^{l_\tau} \left[1 + \delta_{f' s} \delta_{j_{f'} \rho_{\tau'}^\tau} (a^{-1}(\mu_t, \lambda_s) - 1) \right], \quad (6.71)
\end{aligned}$$

iii.) when $n_t \geq 2$,

$$\begin{aligned}
(g')_4^\pm(\mu_\beta, n_t) &= -\sum_{k=0}^{n_t-2} \frac{1}{k!} \frac{e^{\mp \mu_\beta}}{\sinh(\mu_\beta - \mu_e + \eta)^{n_t-k-1}} \frac{d^k}{d(\chi'_4)^k} \{ \sinh^{-1}(\mu_t - \mu_\beta) \\
&\times \prod_{\substack{s=1 \\ s \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_k}^k\}}}^p \prod_{f'=1}^p \prod_{t'=1}^{l_t} [\sinh(\mu_t - \mu_\beta - \eta) \\
&\quad + \delta_{f' s} \delta_{j_{f'} \rho_{t'}^t} (\sinh(\mu_t - \mu_\beta) - \sinh(\mu_t - \mu_\beta - \eta))] \\
&\times \frac{e^{\pm \mu_\beta} b^\pm(\lambda_\alpha, \mu_\beta)}{a(\lambda_\alpha, \mu_\beta)} \prod_{e=1}^k a^{l'_e}(\mu_\beta, \lambda_e) \prod_{t=1}^{k'} \prod_{t'=1}^{l'_t} a^{-1}(\mu_\beta, \lambda_{\rho_{t'}^t}) \prod_{\gamma=p+1}^{\alpha-1} a^{-1}(\mu_\beta, \lambda_\gamma) \\
&\times \prod_{\epsilon=1, \neq t}^{\beta-1} a^{-1}(\mu_\beta, \mu_\epsilon) \left[\prod_{j=p+1}^{n_1} a(\mu_\beta, \lambda_j^{(1)}) - \prod_{l=1}^N a(\mu_\beta, \xi_l) \right] \\
&\times \prod_{\tau=1, \neq t}^k \prod_{\substack{s=1 \\ s \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_k}^k\}}}^p \prod_{f'=1}^p \prod_{\tau'=1}^{l_\tau} \left[1 + \delta_{f' s} \delta_{j_{f'} \rho_{\tau'}^\tau} (a(\mu_\beta, \mu_\tau) - 1) \right] \\
&\times \prod_{\tau=1}^k \prod_{\substack{s=1 \\ s \neq \{\rho_{m_1}^1\}, \dots, \{\rho_{m_k}^k\}}}^p \prod_{f'=1}^p \prod_{\tau'=1}^{l_\tau} \left[1 + \delta_{f' s} \delta_{j_{f'} \rho_{\tau'}^\tau} (a^{-1}(\mu_\beta, \lambda_s) - 1) \right] \Bigg\}_{\chi'_4=0}, \quad (6.72)
\end{aligned}$$

and $\chi'_4 \equiv \chi_2(\mu_\beta) = \sinh(\mu_\beta - \mu_e + \eta)$.

Therefore from (6.27) and (6.50)-(6.51), we have the following theorem:

Theorem 6. *Let the spectral parameters $\{\lambda_k\}$ of the Bethe state $|\Omega_N(\{\lambda_k\}_{(p, n_1)})\rangle$*

be solutions of the BAE (6.7). The scalar products $\mathbb{P}_{n_1}(\{\mu_k\}_{(p,n_1)}, \{\lambda_k\}_{(p,n_1)})$ defined by (6.24) are represented by

$$\begin{aligned}
& \mathbb{P}_{n_1}(\{\mu_k\}_{(p,n_1)}, \{\lambda_k\}_{(p,n_1)}) \\
&= (-1)^{n_1} \sum_{\sigma, \sigma' \in \mathcal{S}_{n_1}} Y_L(\{\mu_{\sigma'(j)}\}, \{\mu_{\sigma'(k)}^{(1)}\}) Y_R(\{\lambda_{\sigma(j)}\}, \{\lambda_{\sigma(k)}^{(1)}\}) \\
&\quad \times \{ \det \mathcal{F}^-(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(p)}; \mu_{\sigma'(1)}, \dots, \mu_{\sigma'(p)}) \\
&\quad \times \det \mathcal{N}_1^-(\lambda_{\sigma(p+1)}, \dots, \lambda_{\sigma(n_1)}; \mu_{\sigma'(p+1)}, \dots, \mu_{\sigma'(n_1)}) \\
&\quad + \sum_{k, l_k, \rho_{m_k}^k} ' \sum_{\substack{j_1, \dots, j_p=1 \\ j_1 \neq \dots \neq j_p}}^p (-1)^{\tau(\sigma'(j_1) \dots \sigma'(j_p))} (A_2^-)_{\sigma(1) \sigma'(j_1)} \dots (A_2^-)_{\sigma(p) \sigma'(j_p)} \\
&\quad \times \det \mathcal{N}_2^-(\lambda_{\sigma(p+1)}, \dots, \lambda_{\sigma(n_1)}; \mu_{\sigma'(p+1)}, \dots, \mu_{\sigma'(n_1)}; \{l_k\}) \\
&\quad + \sum_{k, l_k, \rho_{m_k}^k} '' \det A_3^-(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(p)}; \mu_{\sigma'(1)}, \dots, \mu_{\sigma'(p)}) \\
&\quad \times \det \mathcal{N}_3^-(\lambda_{\sigma(p+1)}, \dots, \lambda_{\sigma(n_1)}; \mu_{\sigma'(p+1)}, \dots, \mu_{\sigma'(n_1)}; \{l_k\}) \\
&\quad + \sum_{k, l_k, \rho_{m_k}^k} ' \sum_{k', l_{k'}, \rho_{m_{k'}}^{k'}} '' \sum_{\substack{j_1, \dots, j_p=1 \\ j_1 \neq \dots \neq j_p}}^p (-1)^{\tau(\sigma'(j_1) \dots \sigma'(j_p))} \\
&\quad \times (A_4^-)_{\sigma(1) \sigma'(j_1)} \dots (A_4^-)_{\sigma(p) \sigma'(j_p)} \\
&\quad \times \det \mathcal{N}_4^-(\lambda_{\sigma(p+1)}, \dots, \lambda_{\sigma(n_1)}; \mu_{\sigma'(p+1)}, \dots, \mu_{\sigma'(n_1)}; \{l_k\}; \{l_{k'}'\}) \}.
\end{aligned} \tag{6.73}$$

Remark: In the derivation of (6.73), the spectral parameters $\{\lambda_i\}$ in the state $|\Omega_N(\{\lambda_j\}_{(p,n_1)})\rangle$ are required to satisfy the BAE (6.7). However, the parameters μ_j ($j = 1, \dots, n_1$) in the dual state $\langle \Omega_N(\{\mu_j\}_{(p,n_1)}) |$ do not need to satisfy the BAE.

On the other hand, if we compute the scalar product by starting from the dual state $\langle \Omega_N(\{\lambda_j\}_{(p,n_1)}) |$, then by using the same procedure, we have

$$\begin{aligned}
& \mathbb{P}_{n_1}^L(\{\lambda_k\}_{(p,n_1)}, \{\mu_j\}_{(p,n_1)}) \\
&= (-1)^{n_1} \sum_{\sigma, \sigma' \in \mathcal{S}_{n_1}} Y_L(\{\lambda_{\sigma(j)}\}, \{\lambda_{\sigma(k)}^{(1)}\}) Y_R(\{\mu_{\sigma'(j)}\}, \{\mu_{\sigma'(k)}^{(1)}\}) \\
&\quad \times G_L^{(n_1)}(\{\mu_{\sigma'(j)}\}_{(p,n_1)}, \{\lambda_{\sigma(k)}\}_{(p,n_1)}),
\end{aligned} \tag{6.74}$$

where the function G_L , compared with G in (6.51), is given by $G_L \equiv \sum_{i=1}^4 (G^+)_i$. In (6.74), we have also assumed that any element of the spectral parameter set $\{\lambda_i\}$ satisfy the BAE.

6.4. Determinant representation of correlation functions

Having obtained the scalar product and the norm, we are now in the position to compute the k-point correlation functions of the model. In general, a l -point

correlation function of the local generators $E_{(k)}^{i,j}$ is defined by

$$\langle E_{(k_1)}^{i_1, j_1} \dots E_{(k_l)}^{i_l, j_l} \rangle \equiv \langle \Omega_N(\{\mu_j\}) | E_{(k_1)}^{i_1, j_1} \dots E_{(k_l)}^{i_l, j_l} | \Omega_N(\{\lambda_j\}) \rangle. \quad (6.75)$$

In principle, by using the theorem 2 and theorem 6, we may compute any correlation function defined by (6.75). As an example, in this subsection, we compute the correlation function associated with two adjacent generators $E_{(\kappa)}^{3,2}$ and $E_{(\kappa+1)}^{2,3}$.

Proposition 7. *If both the Bethe state $|\Omega_N(\{\mu_j\})\rangle$ and the dual Bethe state $\langle \Omega_N(\{\mu_j\})|$ ($\mu_j = \lambda_j, \mu_j$) are eigenstates of the transfer matrix, then the two-point correlation functions associated with the local generators $E_{(\kappa)}^{3,2}$ and $E_{(\kappa+1)}^{2,3}$ can be represented by*

$$\begin{aligned} & \langle \Omega_N(\{\mu_j\})_{(p, n_1)} | E_{(\kappa)}^{3,2} E_{(\kappa+1)}^{2,3} | \Omega_N(\{\lambda_j\})_{(p, n_1)} \rangle \\ &= \sum_{\sigma \in \mathcal{S}_{n_1}} \sum_{\sigma' \in \mathcal{S}_{n_1}} \phi_{\kappa-1}(\{\mu_j\}) \phi_{\kappa+1}^{-1}(\{\lambda_k\}) \sum_{i=1}^p (-1)^{i-1} \frac{b^+(\xi_{\kappa+1}, \lambda_{\sigma(i)})}{a(\xi_{\kappa+1}, \lambda_{\sigma(i)})} \\ & \times \left\{ \sum_{j=i+1}^p \frac{b^+(\xi_{\kappa+1}, \lambda_{\sigma(j)})}{a(\xi_{\kappa+1}, \lambda_{\sigma(j)})} \prod_{l=i+1}^{j-1} \frac{c(\lambda_{\sigma(l)}, \lambda_{\sigma(j)})}{a(\lambda_{\sigma(l)}, \xi_{\kappa})} \prod_{k=i+1, \neq j}^p \frac{c(\lambda_{\sigma(j)}, \lambda_{\sigma(k)})}{a(\lambda_{\sigma(j)}, \lambda_{\sigma(k)})} \right. \\ & \times \prod_{\alpha=1}^N a(\lambda_{\sigma(j)}, \xi_{\alpha}) \mathcal{K} \left(i+1; \lambda_{\sigma(i)}; \{\mu_{\sigma'(d)}\}_{(p, n_1)}; \{\lambda'_{\sigma(f)}\}_{(p, n_1)} \right) \\ & - \prod_{j=i+1}^p \frac{c(\lambda_{\sigma(i)}, \lambda_{\sigma(j)})}{a(\lambda_{\sigma(i)}, \lambda_{\sigma(j)})} \prod_{\alpha=1}^N a(\lambda_{\sigma(i)}, \xi_{\alpha}) \\ & \times \mathcal{K} \left(i+1; \xi_{\kappa+1}; \{\mu_{\sigma'(d)}\}_{(p, n_1)}; \{\lambda^*_{\sigma(f)}\}_{(p, n_1)} \right) \\ & - \sum_{j=i+1}^p \frac{b^+(\lambda_{\sigma(i)}, \lambda_{\sigma(j)})}{a(\lambda_{\sigma(i)}, \lambda_{\sigma(j)})} \prod_{l=i+1}^{j-1} \frac{c(\lambda_{\sigma(l)}, \lambda_{\sigma(j)})}{a(\lambda_{\sigma(l)}, \lambda_{\sigma(i)})} \prod_{k=i+1, \neq j}^p \frac{c(\lambda_{\sigma(j)}, \lambda_{\sigma(k)})}{a(\lambda_{\sigma(j)}, \lambda_{\sigma(k)})} \\ & \times \left. \prod_{\alpha=1}^N a(\lambda_{\sigma(j)}, \xi_{\alpha}) \mathcal{K} \left(i+1; \xi_{\kappa+1}; \{\mu_{\sigma'(d)}\}_{(p, n_1)}; \{\lambda''_{\sigma(f)}\}_{(p, n_1)} \right) \right\}, \quad (6.76) \end{aligned}$$

where $\phi_i(\{\mu_j\}) = \prod_{k=1}^i \prod_{l=1}^n a^{-1}(\mu_l, \xi_k)$, \mathcal{K} is given by

$$\begin{aligned} & \mathcal{K}(e; \delta; \{\mu_{\sigma'(j)}\}; \{\lambda_{\sigma(k)}\}) \\ &= \prod_{i=e}^{n_1} \frac{1}{a(\lambda_{\sigma(i)}, \delta)} P_{n_1}^L(\{\mu_{\sigma'(j)}\}_{(p, n_1)}; \{\lambda_{\sigma(k)}\}_{(p, n_1)}) \\ & - \sum_{j=p+1}^{n_1} \frac{b^-(\lambda_{\sigma(j)}, \delta)}{a(\lambda_{\sigma(j)}, \delta)} \prod_{k=p+1}^{j-1} \frac{c(\lambda_{\sigma(k)}, \lambda_{\sigma(j)})}{c(\lambda_{\sigma(k)}, \delta)} \prod_{l=e, \neq j}^{n_1} \frac{1}{a(\lambda_{\sigma(l)}, \lambda_{\sigma(j)})} \\ & \times P_{n_1}^L(\{\mu_{\sigma'(d)}\}_{(p, n_1)}; \{\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(e)}, \dots, \lambda_{\sigma(j-1)}, \delta, \lambda_{\sigma(j+1)}, \dots, \lambda_{\sigma(n_1)}\}_{(p, n_1)}) \\ & - \sum_{i=e}^p \frac{b^-(\lambda_{\sigma(i)}, \delta)}{a(\lambda_{\sigma(i)}, \delta)} \prod_{k=1}^{i-1} \frac{c(\lambda_{\sigma(k)}, \lambda_{\sigma(i)})}{c(\lambda_{\sigma(k)}, \delta)} \prod_{j=1, \neq i}^p \frac{1}{a(\lambda_{\sigma(j)}, \lambda_{\sigma(i)})} \left[\prod_{l=p+1}^{n_1} \frac{1}{a(\lambda_{\sigma(l)}, \lambda_{\sigma(i)})} \right] \end{aligned}$$

$$\begin{aligned}
& \times P_{n_1}^L(\{\mu_{\sigma'(d)}\}_{(p,n_1)}; \{\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(e)}, \dots, \lambda_{\sigma(i-1)}, \delta, \lambda_{\sigma(i+1)}, \dots, \lambda_{\sigma(n_1)}\}_{(p,n_1)}) \\
& - \sum_{l=p+1}^{n_1} \frac{b^-(\lambda_{\sigma(l)}, \lambda_{\sigma(i)})}{a(\lambda_{\sigma(l)}, \lambda_{\sigma(i)})} \prod_{m=p+1}^{l-1} \frac{c(\lambda_{\sigma(m)}, \lambda_{\sigma(l)})}{c(\lambda_{\sigma(m)}, \lambda_{\sigma(i)})} \prod_{q=p+1, \neq l}^{n_1} \frac{1}{a(\lambda_{\sigma(q)}, \lambda_{\sigma(l)})} \\
& \times P_{n_1}^L(\{\mu_{\sigma'(d)}\}_{(p,n_1)}; \{\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(e)}, \dots, \lambda_{\sigma(i-1)}, \delta, \lambda_{\sigma(i+1)}, \dots, \\
& \quad \lambda_{\sigma(l-1)}, \lambda_{\sigma(i)}, \lambda_{\sigma(l+1)}, \dots, \lambda_{\sigma(n_1)}\}_{(p,n_1)}) \Big] \quad (6.77)
\end{aligned}$$

and the spectral parameters λ', λ^* and λ'' are given by

$$\begin{aligned}
\lambda'_{\sigma(k)} &= \begin{cases} \xi_\kappa & (k=1) \\ \lambda_{\sigma(k-1)} & (2 \leq k \leq i) \\ \lambda_{\sigma(k)} & (i+1 \leq k \leq n_1 \text{ and } k \neq j) \\ \xi_{\kappa+1} & (k=j) \end{cases}, \\
\lambda^*_{\sigma(k)} &= \begin{cases} \xi_\kappa & (k=1) \\ \lambda_{\sigma(k-1)} & (k=2, \dots, i) \\ \lambda_{\sigma(k)} & (i+1 \leq k \leq n_1) \end{cases}, \\
\lambda''_{\sigma(k)} &= \begin{cases} \xi_\kappa & (k=1) \\ \lambda_{\sigma(k-1)} & (2 \leq k \leq i) \\ \lambda_{\sigma(k)} & (i+1 \leq k \leq n_1 \text{ and } k \neq j) \\ \lambda_{\sigma(i)} & (k=j) \end{cases},
\end{aligned}$$

respectively.

Proof. From the definition (6.75), the correlation function is written by

$$\begin{aligned}
& \langle \Omega_N^p(\{\mu_j\})_{(p,n_1)} | E_{(\kappa)}^{3,2} E_{(\kappa+1)}^{2,3} | \Omega_N^p(\{\lambda_k\})_{(p,n_1)} \rangle \\
& = \langle \Omega_N^p(\{\mu_j\})_{(p,n_1)} | \prod_{j=1}^{\kappa-1} t(\xi_j) C_2(\xi_\kappa) B_2(\xi_{\kappa+1}) \prod_{j=\kappa+2}^N t(\xi_j) | \Omega_N^p(\{\lambda_k\})_{(p,n_1)} \rangle \quad (6.78) \\
& = \prod_{j=1}^{n_1} \prod_{k=1}^{\kappa-1} a^{-1}(\mu_j, \xi_k) \prod_{j=1}^{n_1+1} \prod_{k=\kappa+1}^N a^{-1}(\lambda_j, \xi_k) \\
& \quad \times \langle \Omega_N^p(\{\mu_j\})_{(p,n_1)} | C_2(\xi_\kappa) B_2(\xi_{\kappa+1}) | \Omega_N^p(\{\lambda_k\})_{(p,n_1)} \rangle \quad (6.79)
\end{aligned}$$

$$\begin{aligned}
& = \sum_{\sigma \in \mathcal{S}_{n_1}} \sum_{\sigma' \in \mathcal{S}_{n_1}} \phi_{\kappa-1}(\{\lambda_{\sigma(j)}\}) \phi_{\kappa+1}^{-1}(\{\mu_{\sigma'(k)}\}) Y_L(\{\mu_{\sigma'(j)}\}, \{\mu_{\sigma'(k)}^{(1)}\}) \\
& \quad \times Y_R(\{\lambda_{\sigma(j)}\}, \{\lambda_{\sigma(k)}^{(1)}\}) \langle 0 | \prod_{i=p+1}^{\overleftarrow{n_1}} B_1(\mu_{\sigma'(i)}) \prod_{i=1}^{\overleftarrow{p}} B_2(\mu_{\sigma'(i)}) \\
& \quad \times C_2(\xi_\kappa) B_2(\xi_{\kappa+1}) \prod_{i=1}^{\overrightarrow{p}} C_2(\lambda_{\sigma(i)}) \prod_{i=p+1}^{\overrightarrow{n_1}} C_1(\lambda_{\sigma(i)}) | 0 \rangle, \quad (6.80)
\end{aligned}$$

where in (6.78), we have used the theorem 2 and the property: for the q -deformed supersymmetric t - J model with periodic boundary condition, the transfer matrices satisfy the relation $\prod_{i=1}^N t(\lambda_i) = 1$; in (6.79), we have used the theorem 5; and

(6.80), we have used the relation $\prod_{j=1}^n \prod_{k=1}^N a^{-1}(\lambda_j, \xi_k) = 1$, which is from the BAE and the NBAE.

Then with the help of the following commutation relations

$$C_a(\lambda)C_a(\mu) = -c(\lambda, \mu)C_a(\mu)C_a(\lambda), \quad (6.81)$$

$$A_{ab}(\lambda)C_c(\mu) = \frac{r(\lambda - \mu)_{de}^{bc}}{a(\lambda - \mu)}C_e(\mu)A_{ad}(\lambda) + \frac{b^+(\lambda - \mu)}{a(\lambda - \mu)}C_b(\lambda)A_{ac}(\mu), \quad (6.82)$$

$$D(\lambda)C_c(\mu) = \frac{1}{a(\mu, \lambda)}C_c(\mu)D(\lambda) - \frac{b^-(\mu, \lambda)}{a(\mu, \lambda)}C_c(\lambda)D(\mu), \quad (6.83)$$

$$B_a(\lambda)C_b(\mu) = -C_b(\mu)B_a(\lambda) + \frac{b^+(\lambda, \mu)}{a(\lambda, \mu)}[D(\mu)A_{ab}(\lambda) - D(\lambda)A_{ab}(\mu)] \quad (6.84)$$

and the theorem 6, we arrive at this proposition. \square

7. Conclusion and outlook

We have reviewed our recent progress on the construction of the determinant representations of the correlation functions for supersymmetric fermion models via the algebraic Bethe ansatz. The main idea was to simplify the creation (or annihilation) operators and therefore the Bethe state (or its dual state) with the help of the Drinfeld twists. In the F -basis, the creation operators and the Bethe states can be represented in completely symmetric forms. This leads to the scalar products of Bethe states represented by determinants. The determinant representations of the correlation functions were then constructed by means of the scalar products. The determinant representations are useful for analysing asymptotics of time and temperature dependant correlation functions^{53–57} which are important in statistical mechanics and condensed matter physics. They also have applications in algebraic combinatorics and alternating sign matrices.

It would be interesting to generalize the construction of the correlation functions to other integrable models, e.g. the elliptic Z_N Belavin model (for which the symmetric representations of the Bethe states was obtained in³⁵), the supersymmetric U model⁴⁶, the EKS model⁴⁴ and the Hubbard model.

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